Topics on New Keynesian Framework
I. Basic Model and Optimal Policy

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Content[1]:

- Introduction.
- Three-Equation New Keynesian Model.
- Fiscal and Monetary Policy in HANK.
- Intertemporal Keynesian Cross.
- Numerical Methods.

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Outline

Background

- Real business cycle model = neoclassical growth + shocks (on technology or government spending) + endogenous labor choice.
  - The efficiency of business cycles, in contrast with Keynes (1936).
  - The importance of technology shocks as a source of economic fluctuations.
  - The limited role of monetary factors, even abstracting from the existence of a monetary sector in the model.
- Its strong influence among academic researchers notwithstanding, the RBC approach had a very limited impact on central banks and other policy institutions.

Background

- New Keynesian model = RBC + monopolistic competition + nominal rigidities.
  - The prices of goods and inputs are set by private economic agents.
  - Firms are subject to some constraints on the frequency with which they can adjust the prices of the goods and services they sell.
  - Short run non-neutrality of monetary policy as a consequence of the presence of nominal rigidities.
- In this framework, the economy's response to shocks is generally inefficient. Moreover, the non-neutrality of monetary policy resulting from the presence of nominal rigidities makes room for potentially welfare-enhancing interventions by the monetary authority in order to minimize the existing distortions.

Features of New Keynesian Framework

- "New":
  - Based on Walrasian equilibrium, as the market clearing is imposed.
  - Incorporate rational expectation, forward-looking based on mathematical expectation.
  - Behavioral equations describing aggregate variables were thus replaced by FOCs of intertemporal problems facing consumers and firms.
- "Keynesian":
  - A framework to allow demand to partially determine output (supply).
  - Philips curve, i.e., output-inflation trade-off in short run, generated by nominal rigidities.
  - Inefficiency motives policy intervention, due to the monopolistic competitive markets.

Total Utility Function

- Assume a representative infinitely-lived household, seeking to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t), \tag{2.1}$$

where $\beta \in (0,1)$ denotes the discount factor, N_t denotes hours of work or employment, and C_t is a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

with $C_t(i)$ representing the quantity of good i consumed by the household in period t. Assume the existence of a continuum of goods represented by the interval $[0,1]$[2].

[2] Assume that $U_C > 0$, $U_{CC} \leq 0$, $U_N \leq 0$, and $U_{NN} \leq 0$.

Budget Constraint

- The period budget constraint now takes the form

$$\int_0^1 P_t(i)C_t(i)\mathrm{d}i + Q_t B_t \leq B_{t-1} + W_t N_t + T_t, \tag{2.2}$$

  where W_t is the nominal wage, B_t represents purchases of one-period bonds with the price $Q_t = \frac{1}{1+I_t}$, I_t is the nominal interest, and T_t is a lump-sum component of income.
- Additionally, assumed that the household is subject to a solvency constraint that prevents it from engaging in Ponzi-type schemes, by the following constraint:

$$\forall t, \ \lim_{T\to\infty} \mathbb{E}_t(B_T) \geq 0.$$

Optimal Expenditure Allocation

- Define Z_t as the total nominal expenditure, $P_t(i)$ as the price of good i. The optimal allocation of any given expenditure level can be found by solving

$$\max_{C_t(i)} C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

s.t.

$$\int_0^1 P_t(i) C_t(i) \mathrm{d}i \equiv Z_t.$$

- Lagrangian:

$$\mathscr{L} = \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda_t \left(\int_0^1 P_t(i) C_t(i) \mathrm{d}i - Z_t\right).$$

- FOC:

$$\frac{\partial \mathscr{L}}{\partial C_t(i)} : C_t^{\frac{1}{\varepsilon}} C_t(i)^{-\frac{1}{\varepsilon}} = \lambda_t P_t(i), \;\; \forall i \in [0,1],$$

that is, $\forall i, j \in [0,1]$,

$$C_t(i) = C_t(j) \left[\frac{P_t(i)}{P_t(j)}\right]^{-\varepsilon}.$$

Optimal Expenditure Allocation

- The isoelastic demand of good i can be represented as[3]

$$C_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon} C_t,$$

where P_t denote the aggregate price index

$$P_t \equiv \left[\int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}.$$

- **Intuition**: the higher ε, the lower market power.

[3] See appendix 1 for details.

Optimal Expenditure Allocation

- Furthermore, we have

$$\int_0^1 P_t(i)C_t(i)\mathrm{d}i = P_t C_i, \tag{2.3}$$

i.e., total consumption expenditures can be written as the product of the price index times the quantity index.

Intertemporal UMP

- Combining 2.1, 2.2 and 2.3, the intertemporal UMP can be written as

$$\max_{C_t, N_t, B_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t),$$

s.t.

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + T_t.$$

- This is formally identical to the constraint facing households in the single good economy analyzed in previous classical monetary model.

Intertemporal UMP

- Current-value Hamiltonian:

$$\mathscr{H} = U(C_t, N_t) - \lambda_t (P_t C_t + Q_t B_t - B_{t-1} - W_t N_t - T_t).$$

- FOCs:

$$\frac{\partial \mathscr{H}}{\partial C_t} : U_{C,t} = \lambda_t P_t, \tag{2.4}$$

$$\frac{\partial \mathscr{H}}{\partial N_t} : -U_{N,t} = \lambda_t W_t, \tag{2.5}$$

$$\frac{\partial \mathscr{H}}{\partial B_t} : \lambda_t Q_t = \beta \mathbb{E}_t \lambda_{t+1}. \tag{2.6}$$

Intertemporal UMP

- Combining 2.4 and 2.5, we obtain the optimal labor supply decision:

$$-\frac{U_{N,t}}{U_{C,t}} = \frac{W_t}{P_t}.$$

- **Intuition**: $MRS_{C,N}$ is equal to the real wage.
- Combining 2.4 and 2.6, we obtain the Euler equation:

$$Q_t = \beta \mathbb{E}_t \left[\frac{U_{C,t+1}}{U_c, t} \frac{P_t}{P_{t+1}} \right].$$

- Under the assumption of a period utility given by

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi},$$

where $\sigma > 0$, $\sigma \neq 1$, and $\varphi > 0$, then the two equations become

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}, \tag{2.7}$$

$$Q_t = \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right]. \tag{2.8}$$

Optimal Labor Supply

- Log-linearize 2.7, we obtain the log-linearized optimal labor supply decision:

$$\sigma\hat{c}_t + \varphi\hat{n}_t = \hat{w}_t - \hat{p}_t. \tag{2.9}$$

- **Intuition**: Given the marginal utility of consumption (which under the assumptions is a function of consumption only), this equation determines labor supply as a function of the real wage.

Optimal Consumption Path

- Log-linearize 2.8, we have

$$\hat{q}_t = -\sigma \mathbb{E}_t \hat{c}_{t+1} + \sigma \hat{c}_t + \hat{p}_t - \mathbb{E}_t \hat{p}_{t+1}.$$

- Denote $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$ as the (log-linearized) inflation. Since $\hat{i}_t = -\hat{q}_t$, we obtain:

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}). \quad (2.10)$$

- **Intuition**: Given the nominal interest, the intertemporal consumption decision is only determined by the expected inflation.

Production Function

- Assume a continuum of firms indexed by $i \in [0, 1]$. Each firm produces a differentiated good, but they all use an identical technology, represented by the production function

$$Y_t(i) = A_t N_t(i)^{1-\alpha},$$

  where $\alpha \in (0, 1)$, A_t represents the level of technology, assumed to be common to all firms and to evolve exogenously over time.
- All firms face an identical isoelastic demand schedule, and take the aggregate price level P_t and aggregate consumption index C_t as given.

Optimal Labor Demand

- Firms minimize their cost by choosing $N_t(i)$, that is,

$$\min_{N_t(i)} \frac{W_t}{P_t} N_t(i),$$

s.t.

$$Y_t(i) = A_t N_t(i)^{1-\alpha}.$$

Optimal Labor Demand

- Lagrangian:

$$\mathscr{L} = \frac{W_t}{P_t} N_t(i) - \lambda_t [A_t N_t(i)^{1-\alpha} - Y_t(i)].$$

- FOC:

$$\frac{\partial \mathscr{L}}{\partial N_t(i)} : W_t = (1-\alpha)\lambda_t A_t N_t(i)^{-\alpha}.$$

- Denote $MC_t = \lambda_t P_t$ as the nominal marginal cost. Then the FOC becomes

$$W_t = MC_t (1-\alpha) A_t N_t(i)^{-\alpha}.$$

Optimal Price Setting

- Consider the firm's PMP in flexible price system:

$$\max_{P_t(i)} P_t(i)Y_t(i) - W_t(i)N_t(i).$$

- Dividing P_t in this form. By the optimal labor demand decision, when good market is clear, the objective function can be transformed as

$$\begin{aligned}
&\frac{P_t(i)Y_t(i) - W_t(i)N_t(i)}{P_t} \\
= &\left[\frac{P_t(i)}{P_t} - (1-\alpha)\frac{MC_t}{P_t}\right] C_t(i) \\
= &\left[\left[\frac{P_t(i)}{P_t}\right]^{1-\varepsilon} - (1-\alpha)\frac{MC_t}{P_t}\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}\right] C_t.
\end{aligned}$$

Optimal Price Setting

- FOC:

$$(1-\varepsilon)\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}+(1-\alpha)\varepsilon\frac{MC_t}{P_t}\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon-1}=0.$$

- The optimal price:

$$P_t^*(i)=\frac{1-\alpha}{1-\varepsilon^{-1}}MC_t.$$

- **Intuition**: The (nominal) markup $\frac{1-\alpha}{1-\varepsilon^{-1}}$ leads to inefficiently low level of employment and output in the economy due to the monopolistic competition.

Sticky Price Equilibrium

- Following Calvo (1983), each firm may reset its price only with probability $1-\theta$ in any given period, independent of the time elapsed since the last adjustment, i.e., follows an exogenous Poisson process. Thus, each period a measure $1-\theta$ of producers reset their prices, while a fraction θ keep their prices unchanged.
- As a result, the average duration of a price is given by $(1-\theta)^{-1}$. In this context, θ becomes a natural index of price stickiness.

Aggregate Price Dynamics

- Let $S(t) \subset [0,1]$ represent the set of firms not re-optimizing their posted price in period t. The aggregate price index can be written as

$$P_t = \left[\int_{S(t)} P_{t-1}(i)^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$
$$= \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

- Dividing P_{t-1} on both sides and rearranging, we obtain

$$\Pi_t^{1-\varepsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\varepsilon},$$

where $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ is the gross inflation rate between $t-1$ and t.

Aggregate Price Dynamics

- The log-linearized form:

$$\hat{p}_t = \theta \hat{p}_{t-1} + (1-\theta)\hat{p}_t^*, \tag{3.1}$$

or

$$\hat{\pi}_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}).$$

- **Intuition**: Inflation results from the fact that firms re-optimizing in any given period choose a price that differs from the economy's average price in the previous period.

Optimal Price Setting

- With sticky price, the firm must set its price taking account of the risk that it will not be allowed to change its price in the future. That is, firm choose P_t^* to maximize

$$
\begin{aligned}
& [P_t^*(i)Y_t(i) - \Psi_t(i)] \\
& + \theta \mathbb{E}_t[Q_{t,t+1}[P_t^*(i)Y_{t+1|t}(i) - \Psi_{t+1}(i)]] \\
& + \theta^2 \mathbb{E}_t[Q_{t,t+2}[P_t^*(i)Y_{t+2|t}(i) - \Psi_{t+2}(i)]] \\
& + \cdots,
\end{aligned}
$$

where $Q_{t,t+k} \equiv \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$ is the stochastic discount factor for nominal payoffs, $\Psi(\cdot)$ is is the cost function, and $Y_{t+k|t}$ denotes output in period $t+k$ for a firm that last reset its price in period t.

Optimal Price Setting

- Formally, the PMP can be represented as

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t[Q_{t,t+k}[P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t})]],$$

s.t.

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}, \quad k = 0, 1, 2, ...$$

- The log-linearized form of constraint:

$$\hat{y}_{t+k} - \hat{y}_{t+k|t} = \varepsilon(\hat{p}_t^* - \hat{p}_{t+k}) \tag{3.2}$$

Optimal Price Setting

- FOC:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} \left(Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) \right] = 0,$$

where

$$\frac{\partial Y_{t+k|t}}{\partial P_t^*} = -\varepsilon \frac{Y_{t+k|t}}{P_t^*}, \quad k = 0, 1, 2, ...$$

Optimal Price Setting

- Denote nominal marginal cost as $\psi_{t+k|t} \equiv \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}}$, then

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} \left(Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) \right] \\
= &\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} \left(Y_{t+k|t} + (P_t^* - \psi_{t+k|t})(-\varepsilon) \frac{Y_{t+k|t}}{P_t^*} \right) \right] \\
= &\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} Y_{t+k|t} \left[1 + (P_t^* - \psi_{t+k|t}) \frac{-\varepsilon}{P_t^*} \right] \right] \\
= &\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} Y_{t+k|t} \left(1 - \varepsilon + \varepsilon \frac{\psi_{t+k|t}}{P_t^*} \right) \right] = 0.
\end{aligned}
$$

- Solving for the optimal price:

$$P_t^* = \mu \frac{\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t(Q_{t,t+k} Y_{t+k|t} \psi_{t+k|t})}{\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t(Q_{t,t+k} Y_{t+k|t})}, \tag{3.3}$$

  where $\mu \equiv \frac{\varepsilon}{\varepsilon - 1}$.
- In the case of flexible price, i.e., $\theta = 0$, 3.1 becomes $P_t^* = \mu \psi_{t|t}$. Henceforth, μ is referred to as the desired or frictionless markup.
- **Intuition**: The optimal price equals μ multiplied by the ratio of the expected future marginal cost to the expected future output.

- Denote $MC_t^r = \frac{\psi_t}{P_t}$ as the real marginal cost. Consider the log-linearized form of 3.3[4]:

$$\hat{p}_t^* = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t(\hat{mc}^r_{t+k|t} + \hat{p}_{t+k}). \tag{3.4}$$

- The forward-looking equation:

$$\hat{p}_t^* = (1-\beta\theta)(\hat{mc}^r_{t|t} + \hat{p}_t) + \beta\theta\mathbb{E}_t\hat{p}^*_{t+1}. \tag{3.5}$$

- **Intuition**: Firms that are allowed to reset the price do so as a weighted average over the expected future nominal marginal cost.

[4]See appendix 2 for details.

Inflation Dynamics

- Substituting 3.5 into 3.1, we obtain

$$\begin{aligned}
\hat{p}_t &= \theta\hat{p}_{t-1} + (1-\theta)\hat{p}_t^* \\
&= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)(\hat{mc}^r_{t|t} + \hat{p}_t) + (1-\theta)\beta\theta\mathbb{E}_t\hat{p}^*_{t+1} \\
&= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)(\hat{mc}^r_{t|t} + \hat{p}_t) + (1-\theta)\beta\theta\frac{\mathbb{E}_t\hat{p}_{t+1} - \theta\hat{p}_t}{1-\theta} \\
&= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)(\hat{mc}^r_{t|t} + \hat{p}_t) + \beta\theta\mathbb{E}_t\hat{p}_{t+1} - \beta\theta^2\hat{p}_t.
\end{aligned}$$

- Subtracting by $(1-\theta)\hat{p}_t$ and rearranging yields

$$\begin{aligned}
\theta\hat{\pi}_t &\equiv \theta(\hat{p}_t - \hat{p}_{t-1}) \\
&= (1-\theta)(1-\beta\theta)\hat{mc}^r_{t|t} + \beta\theta(\mathbb{E}_t\hat{p}_{t+1} - \hat{p}_t) \\
&= (1-\theta)(1-\beta\theta)\hat{mc}^r_{t|t} + \beta\theta\mathbb{E}_t\hat{\pi}_{t+1}.
\end{aligned}$$

Inflation Dynamics

- The forward-looking equation for inflation:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta} \hat{mc}^r_{t|t}.$$

- **Intuition**: If demand increases, then $Y \uparrow$, $N \uparrow$, $\frac{W}{P} \uparrow$, $MC^r \uparrow$. This triggers a rise in current inflation.

Good Market Clearing

- Market clearing in the goods market requires

$$Y_t(i) = C_t(i), \quad \forall i \in [0,1].$$

- Letting aggregate output be defined as $Y_t \equiv \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}}$ it follows that $Y_t = C_t$, i.e., $\hat{y}_t = \hat{c}_t$.
- Combining with 2.10, we obtain the equilibrium condition of good market clearing:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}). \tag{4.1}$$

Labor Market Clearing

- Market clearing in the labor market requires

$$N_t = \int_0^1 N_t(i) \mathrm{d}i.$$

- Combining with the production function, we obtain

$$\begin{aligned} N_t &= \int_0^1 \left[\frac{Y_t(i)}{A_t} \right]^{\frac{1}{1-\alpha}} \mathrm{d}i \\ &= \left(\frac{Y_t}{A_t} \right)^{\frac{1}{1-\alpha}} \int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\frac{\varepsilon}{1-\alpha}} \mathrm{d}i. \end{aligned}$$

- Taking the square $1 - \alpha$ yields

$$N_t^{1-\alpha} = \frac{Y_t}{A_t} \left[\int_0^1 \left[\frac{P_t(i)}{P_t} \right]^{-\frac{\varepsilon}{1-\alpha}} \mathrm{d}i \right]^{1-\alpha}.$$

Labor Market Clearing

- Denote $D_t = \left[\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon}{1-\alpha}} \mathrm{d}i\right]^{1-\alpha}$ measuring price (output) dispersion across firms, we have

$$Y_t = A_t N_t^{1-\alpha} D_t^{-1}.$$

- **Intuition**: If $D_t > 1$, then inefficiency in output level exists.
- The log-linearized form[5]:

$$\hat{y}_t = \hat{a}_t + (1-\alpha)\hat{n}_t. \tag{4.2}$$

[5]We can prove that $d_t = \ln D_t$ is equal to zero up to a first-order approximation in a neighborhood of $\pi = 0$, see appendix 3 for details.

Average Real Marginal Cost

- In firm's CMP, we have known that

$$MC_t^r = \frac{W_t}{P_t} \frac{1}{A_t(1-\alpha)N_t^{-\alpha}}.$$

- The log-linearized form:

$$\begin{aligned}\hat{mc}_t^r &= \hat{w}_t - \hat{p}_t - (\hat{a}_t - \alpha\hat{n}_t) \\ &= \hat{w}_t - \hat{p}_t - \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t).\end{aligned}$$

Average Real Marginal Cost

- For firms that set price at t and remains it at $t+k$:

$$
\begin{aligned}
\hat{mc}^r_{t+k|t} &= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \\
&= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \\
&= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k}) - \frac{\alpha}{1-\alpha}(\hat{y}_{t+k} - \hat{y}_{t+k|t}) \\
&= \hat{mc}^r_{t+k} - \frac{\alpha\varepsilon}{1-\alpha}(\hat{p}^*_t - \hat{p}_{t+k}), \qquad (4.3)
\end{aligned}
$$

where the last line uses 3.2.

Optimal Price with ARMC

- Substituting 4.3 into 3.4, we have

$$
\begin{aligned}
p_t^* &= \Gamma(\hat{mc}_{t+k|t}^r + \hat{p}_{t+k}) \\
&= \Gamma\left[\hat{mc}_{t+k}^r - \frac{\alpha\varepsilon}{1-\alpha}(\hat{p}_t^* - \hat{p}_{t+k}) + \hat{p}_{t+k}\right] \\
&= -\frac{\alpha\varepsilon}{1-\alpha}\hat{p}_t^* + \Gamma\left(\hat{mc}_{t+k}^r + \Theta^{-1}\hat{p}_{t+k}\right),
\end{aligned}
$$

  where $\Gamma(\cdot) \equiv (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t(\cdot)$, and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\varepsilon} \leq 1$.
- Solving for the optimal price in terms of average real marginal cost:

$$
\begin{aligned}
\hat{p}_t^* &= \Gamma(\Theta\hat{mc}_{t+k}^r + \hat{p}_{t+k}) \\
&= (1-\beta\theta)(\Theta\hat{mc}_t^r + \hat{p}_t) + \beta\theta\mathbb{E}_t\hat{p}_{t+1}^*. \qquad (4.4)
\end{aligned}
$$

Inflation Dynamics with Real Marginal Cost

- Similarly, substituting 4.4 into 3.1, then subtracting by $(1-\theta)\hat{p}_t$ and arranging yields

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \lambda \hat{mc}_t^r, \tag{4.5}$$

where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$.

- Solving forward, we obtain

$$\hat{\pi}_t = \lambda \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \hat{mc}_{t+k}^r.$$

- **Intuition**: Inflation results from the aggregate consequences of purposeful price-setting decisions by firms, which adjust their prices in light of current and anticipated cost conditions.

Output Gap with Real Marginal Cost

- Re-consider the log-linearized form of average real marginal cost, using 2.9 and 4.2 yields

$$
\begin{aligned}
\hat{mc}_t^r &= (\hat{w}_t - \hat{p}_t) - (\hat{a}_t - \alpha\hat{n}_t) \\
&= (\sigma\hat{y}_t + \varphi\hat{n}_t) - \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t) \\
&= \sigma\hat{y}_t + \frac{\varphi}{1-\alpha}(\hat{y}_t - \hat{a}_t) - \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t) \\
&= \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\hat{y}_t - \frac{\varphi+1}{1-\alpha}\hat{a}_t. \qquad (4.6)
\end{aligned}
$$

Output Gap with Real Marginal Cost

- Under flexible prices the real marginal cost is constant ($MC_t^r = \mu^{-1}$), thus $\hat{mc}_t^r = 0$.
- Defining the natural level of output, denoted by y_t^n, as the equilibrium level of output under flexible prices:

$$\hat{y}_t^n = \frac{\varphi + 1}{\varphi + \alpha + \sigma - \alpha\sigma}\hat{a}_t.$$

- Then we have

$$\hat{mc}_t^r = \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha}\right)\tilde{y}_t, \tag{4.7}$$

where $\tilde{y}_t \equiv \hat{y}_t - \hat{y}_t^n$ is denoted as the output gap, the deviation between the real output and the natural level of output.

New Keynesian Philips Curve

- Substituting 4.7 into 4.5, we obtain the New Keynesian Philips Curve (*abbr.* NKPC):

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t, \tag{4.8}$$

  where $\kappa \equiv \lambda \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) > 0$.

Dynamic IS Curve

- Rewriting 4.1 in terms of the output gap as

$$\tilde{y}_t = \mathbb{E}_t(\hat{y}^n_{t+1} - \hat{y}^n_t) + \mathbb{E}_t\tilde{y}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}).$$

- Define the natural rate of interest $\hat{r}^n_t \equiv \sigma\mathbb{E}_t(\hat{y}^n_{t+1} - \hat{y}^n_t)$. Then rewriting the equation in terms of $\hat{r}^n_t$, we obtain the Dynamic IS Curve (*abbr.* DISC):

$$\tilde{y}_t = -\sigma^{-1}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{r}^n_t) + \mathbb{E}_t\tilde{y}_{t+1}. \quad (4.9)$$

Dynamic IS Curve

- Under the assumption that the effects of nominal rigidities vanish asymptotically, $\lim_{T\to\infty} \mathbb{E}_t \tilde{y}_{t+T} = 0$ In that case one can solve 4.9 forward to yield

$$\tilde{y}_t = -\sigma^{-1} \sum_{k=0}^{\infty} \mathbb{E}_t(\hat{r}_{t+k} - \hat{r}^n_{t+k}),$$

  where $\hat{r}^n_t \equiv \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}$ is the (log-linearized) expected real return on a one period bond (i.e., the real interest rate).
- **Intuition**: The output gap is proportional to the sum of current and anticipated deviations between the real interest rate and its natural counterpart.

Summary: Non-Policy Block

- Three core variables: $\{\tilde{y}_t\}$, $\{\hat{\pi}_t\}$, and $\{\hat{i}_t\}$.
- The NKPC determines inflation given a path for the output gap,

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t.$$

- The DISC determines output gap given a path for the (exogenous) natural rate and the actual real rate,

$$\tilde{y}_t = -\sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^n) + \mathbb{E}_t \tilde{y}_{t+1}.$$

Summary: Non-Policy Block

- In order to close the model, we need to supplement these two equations with one or more equations determining how the nominal interest rate $\widehat{i}_t$ evolves over time, i.e., with a description of how the monetary policy is conducted.
  - In New Keynesian models, with the sticky price setting, the equilibrium path of real variables cannot be determined independently of monetary policy. In other words, monetary policy is non-neutral.

The system under an Interest Rate Rule

- Consider the equilibrium under a simple interest rate rule of the form

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + v_t, \tag{5.1}$$

where v_t is an exogenous (possibly stochastic) component with zero mean. Assume ϕ_π and ϕ_y are non-negative coefficients, chosen by the monetary authority.

- Combining 5.1 with 4.9, the DISC becomes

$$\tilde{y}_t = -\sigma^{-1}[\phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t - \mathbb{E}_t \hat{\pi}_{t+1} - (\hat{r}^n_t - v_t)] + \mathbb{E}_t \tilde{y}_{t+1}.$$

The system under an Interest Rate Rule

- Rewriting the NKPC and DISC:

$$\begin{aligned}(1+\sigma^{-1}\phi_y)\tilde{y}_t + \sigma^{-1}\phi_\pi\hat{\pi}_t &= \mathbb{E}_t\tilde{y}_{t+1} + \sigma^{-1}\mathbb{E}_t\hat{\pi}_{t+1} + \sigma^{-1}(\hat{r}_t^n - v_t),\\ -\kappa\tilde{y}_t + \qquad \hat{\pi}_t &= \qquad \beta\mathbb{E}_t\hat{\pi}_{t+1}.\end{aligned}$$

- The matrix form[6]:

$$\begin{bmatrix}\tilde{y}_y\\ \hat{\pi}_t\end{bmatrix} = \mathbf{A}_T\begin{bmatrix}\mathbb{E}_t\tilde{y}_{t+1}\\ \mathbb{E}_t\hat{\pi}_{t+1}\end{bmatrix} + \mathbf{B}_T(\hat{r}_t^n - v_t), \tag{5.2}$$

where $\Omega \equiv (\sigma + \phi_y + \kappa\phi_\pi)^{-1}$, $\mathbf{A}_T \equiv \Omega\begin{bmatrix}\sigma & 1-\beta\phi_\pi\\ \kappa\sigma & \kappa+\beta(\sigma+\phi_y)\end{bmatrix}$, and $\mathbf{B}_T \equiv \Omega\begin{bmatrix}1\\ \kappa\end{bmatrix}$.

[6] See appendix 4 for details.

The system under an Interest Rate Rule

- By the Blanchard-Kahn condition, given that both the $\tilde{y}_t$ and $\hat{\pi}_t$ are nonpredetermined variables, the solution to 5.2 is locally unique, if and only if, $\mathbf{A}_T$ has both eigenvalues within the unit circle.
- Under the assumption of non-negative coefficients (ϕ_π, ϕ_y) it can be shown that a necessary and sufficient condition for uniqueness is given by

$$\kappa(\phi_\pi - 1) + (1-\beta)\phi_y > 0, \tag{5.3}$$

which is assumed to hold, unless stated otherwise[7].

[7] See appendix 4 for details.

Monetary Policy Shock

- Assume an AR(1) process of $\{v_t\}$, that is, $v_t = \rho_v v_{t-1} + \varepsilon_t^v$, $\rho_v \in [0, 1)$.
- A positive (negative) realization of ε_t^v should be interpreted as a contractionary (expansionary) monetary policy shock, leading to a rise (decline) in the nominal interest rate, given $\hat{\pi}_t$ and $\tilde{y}_t$.
- Because the natural rate of interest (output) is not affected by monetary shocks, $\forall t$, $\hat{r}_t^n = \hat{y}_t^n = 0$ is set for the purposes of the present exercise.

Monetary Policy Shock

- Using undetermined coefficient method, guess that

$$\tilde{y}_t = \psi_{yv} v_t,$$
$$\hat{\pi}_t = \psi_{\pi v} v_t,$$

where ψ_{yv} and $\psi_{\pi v}$ are coefficients to be determined.

- Verification[8]:

$$\tilde{y}_t = -(1 - \beta\rho_v)\Lambda_v v_t,$$
$$\hat{\pi}_t = -\kappa\Lambda_v v_t,$$

where $\Lambda_v \equiv [(1 - \beta\rho_v)[\sigma(1 - \rho_v) + \phi_y] + \kappa(\phi_\pi - \rho_v)]^{-1}$.

- Since 5.3 is satisfied, it can be easily shown that $\Lambda_v > 0$. Then we have $-(1 - \beta\rho_v)\Lambda_v < 0$, and $-\kappa\Lambda_v < 0$.
- **Impulse responses**: If $v_t \uparrow$, then $\hat{\pi}_t \downarrow$, $\tilde{y}_t \downarrow$, and $\hat{y}_t \downarrow$.

[8] See appendix 5 for details.

Monetary Policy Shock

- Similarly, we can solve the form of (log-linearized) real interest $\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}$ as[9]

$$\hat{r}_t = \sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v v_t,$$

thus we have $\sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v > 0$.

- Hence,

$$\hat{i}_t = [\sigma(1-\rho_v)(1-\beta\rho_v) - \kappa\rho_v]\Lambda_v v_t.$$

- **Impulse responses**: If $v_t \uparrow$, then $\hat{r}_t \uparrow$. But if the persistence of the monetary policy shock ρ_v is sufficiently high, $\hat{i}_t \downarrow$.
- **Intuition**: The downward adjustment in the nominal rate induced by the decline in inflation and the output gap more than offsetting the direct effect of a higher v_t.

[9] See appendix 5 for details.

Technology Shock

- Assume an AR(1) process of $\{a_t\}$, that is, $\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_t^a$, $\rho_a \in [0, 1]$, and $\{\varepsilon_t^a\}$ is a zero mean white noise process.
- By the definition of the natural rate of interest,

$$\hat{r}_t^n \equiv \sigma \mathbb{E}_t(\hat{y}_{t+1}^n - \hat{y}_t^n) = \sigma \psi_{ya}^n \mathbb{E}_t(\hat{a}_{t+1}^n - \hat{a}_t^n) = -\sigma \psi_{ya}^n (1 - \rho_a) \hat{a}_t,$$

  where $\psi_{ya}^n \equiv \frac{\sigma+1}{\varphi+\alpha+\sigma-\alpha\sigma}$ due to the definition of $\hat{y}_t^n$.
- Turning off monetary shocks, i.e., $\forall t$, $v_t = 0$.

Technology Shock

- Using undetermined coefficient method, guess that

$$\tilde{y}_t = \psi_{yr}\hat{r}_t^n,$$
$$\hat{\pi}_t = \psi_{\pi r}\hat{r}_t^n,$$

  where ψ_{yv} and $\psi_{\pi v}$ are coefficients to be determined.
- Since $\hat{r}_t^n$ enters the system 5.2 in a way symmetric to v_t but with an opposite sign, we obtain $\psi_{yr} = -\psi_{yv}$, and $\psi_{\pi r} = -\psi_{\pi v}$.
- Similarly we have

$$\tilde{y}_t = (1-\beta\rho_a)\Lambda_a\hat{r}_t^n = -\sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a\hat{a}_t,$$
$$\hat{\pi}_t = \kappa\Lambda_a\hat{r}_t^n = -\sigma\psi_{ya}^n(1-\rho_a)\kappa\Lambda_a\hat{a}_t,$$

  where $\Lambda_a \equiv [(1-\beta\rho_a)[\sigma(1-\rho_a)+\phi_y]+\kappa(\phi_\pi-\rho_a)]^{-1} > 0$.
- **Impulse responses**: If $\rho_a < 1$ and $a_t \uparrow$, then $\hat{\pi}_t \downarrow$, and $\tilde{y}_t \downarrow$.

Technology Shock

- Then consider the output and employment:

$$\hat{y}_t \equiv \hat{y}_t^n + \tilde{y}_t = \psi_{ya}^n[1 - \sigma(1-\rho_a)(1-\beta\rho_a)\Lambda_a]\hat{a}_t,$$
$$(1-\alpha)\hat{n}_t \equiv \hat{y}_t - \hat{a}_t = [(\psi_{ya}^n - 1) - \sigma\psi_{ya}^n(1-\rho_a)(1-\beta\rho_a)\Lambda_a]\hat{a}_t.$$

- The sign of the response of output and employment to a positive technology shock is in general ambiguous, depending on the configuration of parameter values, including the interest rate rule coefficients.
- In the baseline calibration, $\sigma = 1$, which implies $\psi_{ya}^n = 1$. In that case, if $a_t \uparrow$, then $\hat{n}_t \downarrow$. Consistency with much empirical evidence.

Social Optimality

- Consider the social planner's problem:

$$\max_{C_t(i), N_t(i)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t),$$

s.t.

$$C_t(i) = A_t N_t(i)^{1-\alpha},$$

$$N_t = \int_0^1 N_t(i) \mathrm{d}i.$$

Social Optimality

- The associated optimality conditions:

$$C_t(i) = C_t, \quad \forall i \in [0,1];$$
$$N_t(i) = N_t, \quad \forall i \in [0,1];$$
$$-\frac{U_{n,t}}{U_{c,t}} = MPN_t,$$

  where $MPN_t \equiv (1-\alpha)A_t N_t^{-\alpha}$ denotes the economy's average marginal product of labor.

- Thus, it is optimal to produce and consume the same quantity of all goods and to allocate the same amount of labor to all firms, and $\forall t$, $MRS_t = MPN_t$ is also required.

Sources of Suboptimality

The basic New Keynesian model is characterized by two distortions:

- The presence of market power in goods markets, exercised by monopolistically competitive firms.
- The assumption of infrequent adjustment of prices by firms due to the sticky price.

Optimal Monetary Policy

- In the case where there are no inherited relative price distortions, i.e., $P_{-1}(i) = P_{-1}$, the efficient allocation can be attained by a policy that stabilizes marginal costs at a level consistent with firms' desired markup, given the prices in place. As a result, $P_t^* = P_{t-1}$, and, hence, $\forall t$, $P_t = P_{t-1}$.
- The optimal monetary policy requires that $\forall t$, $\tilde{y} = 0$, and $\hat{\pi} = 0$. By 4.9, we have

$$\hat{i}_t = \hat{r}_t^n.$$

Taylor Rule

- Consider the following interest rule, in the spirit of Taylor (1993),

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + v_t,$$

  where $v_t \equiv \phi_y \hat{y}_t^n$.

- The resulting equilibrium dynamics are thus identical to those of the interest rate rule analyzed in previous content, with v_t now reinterpreted as a driving force proportional to the deviations of natural output from steady state, instead of an exogenous monetary policy shock.

Calibration

- This table displays some statistics for four different calibrations of Taylor rule, corresponding to alternative configurations for ϕ_π and ϕ_y.
- $\sigma(\tilde{y})$ and $\sigma(\pi)$ show the implied standard deviations of the output gap and (annualized) inflation, both expressed in percent terms, as well as the welfare losses resulting from the associated deviations from the efficient allocation, expressed as a fraction of steady state consumption.

Taylor Rule				
ϕ_π	1.5	1.5	5	1.5
ϕ_y	0.125	0	0	1
$\sigma(\tilde{y})$	0.55	0.28	0.04	1.40
$\sigma(\pi)$	2.60	1.33	0.21	6.55
welfare loss	0.30	0.08	0.002	1.92

Calibration

Implications:

- Versions of the rule that involve a systematic response to output variations generate larger fluctuations in the output gap and inflation and, hence, larger welfare losses.
- The smallest welfare losses are attained when the monetary authority responds to changes in inflation only. Those losses (as well as the underlying fluctuations in the output gap and inflation) become smaller as the strength of that response increases.

Results:

- A simple Taylor-type rule that responds aggressively to movements in inflation can approximate arbitrarily well the optimal policy in the basic New Keynesian model.

Summary

The three equations describing the basic New Keynesian model:

- New Keynesian Philips Curve:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t.$$

- Dynamic IS Curve:

$$\tilde{y}_t = -\sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^n) + \mathbb{E}_t \tilde{y}_{t+1}.$$

- Taylor Rule:

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + v_t.$$

Questions are welcome.