New Keynesian Model with Calvo Pricing

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1 Households

Assume a representative infinitely-lived household, seeking to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t), \tag{1}$$

where $\beta \in (0, 1)$ denotes the discount factor, N_t denotes hours of work or employment, and C_t is a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

with $C_t(i)$ representing the quantity of good i consumed by the household in period t. Assume the existence of a continuum of goods represented by the interval $[0, 1]$[1].

The period budget constraint now takes the form

$$\int_0^1 P_t(i)C_t(i)\mathrm{d}i + Q_t B_t \leq B_{t-1} + W_t N_t + T_t, \tag{2}$$

where W_t is the nominal wage, B_t represents purchases of one-period bonds with the price $Q_t = \frac{1}{1+I_t}$, I_t is the nominal interest, and T_t is a lump-sum component of income.

Additionally, assumed that the household is subject to a solvency constraint that prevents it from engaging in Ponzi-type schemes, by the following constraint:

$$\forall t, \ \lim_{T \to \infty} \mathbb{E}_t(B_T) \geq 0.$$

Define Z_t as the total nominal expenditure, $P_t(i)$ as the price of good i. The optimal allocation of any given expenditure level can be found by solving

$$\max_{C_t(i)} C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

s.t.

$$\int_0^1 P_t(i)C_t(i)\mathrm{d}i \equiv Z_t.$$

Lagrangian:

$$\mathcal{L} = \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}} - \lambda_t \left(\int_0^1 P_t(i)C_t(i)\mathrm{d}i - Z_t \right).$$

[1] Assume that $U_C > 0$, $U_{CC} \leq 0$, $U_N \leq 0$, and $U_{NN} \leq 0$.

FOC:
$$\frac{\partial \mathcal{L}}{\partial C_t(i)} : C_t^{\frac{1}{\varepsilon}} C_t(i)^{-\frac{1}{\varepsilon}} = \lambda_t P_t(i),\ \forall i \in [0,1],$$

that is, $\forall i, j \in [0,1]$,
$$C_t(i) = C_t(j) \left[\frac{P_t(i)}{P_t(j)}\right]^{-\varepsilon}.$$

By the definition of the consumption index, we can derive that

$$\begin{aligned}
C_t^{\frac{\varepsilon-1}{\varepsilon}} &\equiv \int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \\
&= \int_0^1 \left[C_t(j) \left[\frac{P_t(i)}{P_t(j)}\right]^{-\varepsilon}\right]^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \\
&= C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} P_t(j)^{\varepsilon-1} \int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i \\
&\equiv C_t(j)^{\frac{\varepsilon-1}{\varepsilon}} P_t(j)^{\varepsilon-1} P_t^{1-\varepsilon}.
\end{aligned}$$

Taking the square of $\frac{\varepsilon}{\varepsilon-1}$ yields

$$C_t = C_t(j) P_t(j)^{\varepsilon} P_t^{-\varepsilon}.$$

Then the isoelastic demand of good i can be represented as

$$C_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon} C_t,$$

where P_t denote the aggregate price index

$$P_t \equiv \left[\int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}.$$

Furthermore, we have
$$\int_0^1 P_t(i) C_t(i) \mathrm{d}i = P_t C_i, \tag{3}$$
i.e., total consumption expenditures can be written as the product of the price index times the quantity index.

Next we consider the dynamic problem. Combine (1), (2) and (3), the intertemporal UMP can be written as

$$\max_{C_t, N_t, B_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t),$$

s.t.
$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + T_t.$$

Current-value Hamiltonian:

$$\mathcal{H} = U(C_t, N_t) - \lambda_t (P_t C_t + Q_t B_t - B_{t-1} - W_t N_t - T_t).$$

FOCs:

$$\frac{\partial \mathcal{H}}{\partial C_t} : U_{C,t} = \lambda_t P_t, \tag{4}$$

$$\frac{\partial \mathcal{H}}{\partial N_t} : -U_{N,t} = \lambda_t W_t, \tag{5}$$

$$\frac{\partial \mathcal{H}}{\partial B_t} : \lambda_t Q_t = \beta \mathbb{E}_t \lambda_{t+1}. \tag{6}$$

Combine (4) and (5), we obtain the optimal labor supply decision:

$$-\frac{U_{N,t}}{U_{C,t}} = \frac{W_t}{P_t}.$$

Combine (4) and (6), we obtain the Euler equation:

$$Q_t = \beta \mathbb{E}_t \left[\frac{U_{C,t+1}}{U_c, t} \frac{P_t}{P_{t+1}} \right].$$

Under the assumption of a period utility given by

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi},$$

where $\sigma > 0$, $\sigma \neq 1$, and $\varphi > 0$, then the two equations become

$$C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}, \tag{7}$$

$$Q_t = \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right]. \tag{8}$$

Log-linearize (7), we obtain the log-linearized optimal labor supply decision:

$$\sigma \hat{c}_t + \varphi \hat{n}_t = \hat{w}_t - \hat{p}_t. \tag{9}$$

This shows that, given the marginal utility of consumption (which under the assumptions is a function of consumption only), this equation determines labor supply as a function of the real wage.

Log-linearize (8), we have

$$\hat{q}_t = -\sigma \mathbb{E}_t \hat{c}_{t+1} + \sigma \hat{c}_t + \hat{p}_t - \mathbb{E}_t \hat{p}_{t+1}.$$

Denote $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$ as the (log-linearized) inflation. Since $\hat{i}_t = -\hat{q}_t$, we obtain:

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}). \tag{10}$$

This shows that, given the nominal interest, the intertemporal consumption decision is only determined by the expected inflation.

2 Firms

Assume a continuum of firms indexed by $i \in [0,1]$. Each firm produces a differentiated good, but they all use an identical technology, represented by the production function

$$Y_t(i) = A_t N_t(i)^{1-\alpha},$$

where $\alpha \in (0,1)$, A_t represents the level of technology, assumed to be common to all firms and to evolve exogenously over time. All firms face an identical isoelastic demand schedule, and take the aggregate price level P_t and aggregate consumption index C_t as given.

Consider the optimal labor demand for firms. One minimizes its cost by choosing $N_t(i)$, that is,

$$\min_{N_t(i)} \frac{W_t}{P_t} N_t(i),$$

s.t.

$$Y_t(i) = A_t N_t(i)^{1-\alpha}.$$

Lagrangian:

$$\mathcal{L} = \frac{W_t}{P_t} N_t(i) - \lambda_t [A_t N_t(i)^{1-\alpha} - Y_t(i)].$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial N_t(i)} : W_t = (1-\alpha)\lambda_t A_t N_t(i)^{-\alpha}.$$

Denote $MC_t = \lambda_t P_t$ as the nominal marginal cost. Then the FOC becomes

$$W_t = MC_t (1-\alpha) A_t N_t(i)^{-\alpha}.$$

In flexible price system, the firm's PMP can be represented as

$$\max_{P_t(i)} P_t(i) Y_t(i) - W_t(i) N_t(i).$$

Firstly we divide the objective function by P_t. By the optimal labor demand decision, when good market is clear, the objective function can be transformed as

$$\begin{aligned} & \frac{P_t(i) Y_t(i) - W_t(i) N_t(i)}{P_t} \\ = & \left[\frac{P_t(i)}{P_t} - (1-\alpha) \frac{MC_t}{P_t} \right] C_t(i) \\ = & \left[\left[\frac{P_t(i)}{P_t} \right]^{1-\varepsilon} - (1-\alpha) \frac{MC_t}{P_t} \left[\frac{P_t(i)}{P_t} \right]^{-\varepsilon} \right] C_t. \end{aligned}$$

FOC:

$$(1-\varepsilon) \left[\frac{P_t(i)}{P_t} \right]^{-\varepsilon} + (1-\alpha)\varepsilon \frac{MC_t}{P_t} \left[\frac{P_t(i)}{P_t} \right]^{-\varepsilon-1} = 0.$$

Thus, the optimal price becomes

$$P_t^*(i) = \frac{1-\alpha}{1-\varepsilon^{-1}} MC_t.$$

In this form, the (nominal) markup $\frac{1-\alpha}{1-\varepsilon^{-1}}$ leads to inefficiently low level of employment and output in the economy due to the monopolistic competition.

Then we turn to sticky price system. Following Calvo (1983), each firm may reset its price only with probability $1-\theta$ in any given period, independent of the time elapsed since the last adjustment, i.e., follows an exogenous Poisson process. Thus, each period a measure $1-\theta$ of producers reset their prices, while a fraction θ keep their prices unchanged. As a result, the average duration of a price is given by $(1-\theta)^{-1}$. In this context, θ becomes a natural index of price stickiness.

Let $S(t) \subset [0,1]$ represent the set of firms not re-optimizing their posted price in period t. The aggregate price index can be written as

$$\begin{aligned} P_t &= \left[\int_{S(t)} P_{t-1}(i)^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \\ &= \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}. \end{aligned}$$

Divide P_{t-1} on both sides and rearranging, we obtain

$$\Pi_t^{1-\varepsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\varepsilon},$$

where $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ is the gross inflation rate between $t-1$ and t.

The log-linearized form:

$$\hat{p}_t = \theta\hat{p}_{t-1} + (1-\theta)\hat{p}_t^*, \tag{11}$$

or

$$\hat{\pi}_t = (1-\theta)(\hat{p}_t^* - \hat{p}_{t-1}).$$

This implies that, inflation results from the fact that firms re-optimizing in any given period choose a price that differs from the economy's average price in the previous period.

With sticky price, the firm must set its price taking account of the risk that it will not be allowed to change its price in the future. That is, firm choose P_t^* to maximize

$$\begin{aligned} &[P_t^*(i)Y_t(i) - \Psi_t(i)] \\ &+ \theta\mathbb{E}_t[Q_{t,t+1}[P_t^*(i)Y_{t+1|t}(i) - \Psi_{t+1}(i)]] \\ &+ \theta^2\mathbb{E}_t[Q_{t,t+2}[P_t^*(i)Y_{t+2|t}(i) - \Psi_{t+2}(i)]] \\ &+ \cdots, \end{aligned}$$

where $Q_{t,t+k} \equiv \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$ is the stochastic discount factor for nominal payoffs, $\Psi(\cdot)$ is is the cost function, and $Y_{t+k|t}$ denotes output in period $t+k$ for a firm that last reset its price in period t.

Formally, the PMP can be represented as

$$\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t[Q_{t,t+k}[P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t})]],$$

s.t.

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}, \quad k = 0, 1, 2, ...$$

The log-linearized form of constraint:

$$\hat{y}_{t+k} - \hat{y}_{t+k|t} = \varepsilon(\hat{p}_t^* - \hat{p}_{t+k}) \tag{12}$$

FOC:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} \left(Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) \right] = 0,$$

where

$$\frac{\partial Y_{t+k|t}}{\partial P_t^*} = -\varepsilon \frac{Y_{t+k|t}}{P_t^*}, \quad k = 0, 1, 2, ...$$

Denote nominal marginal cost as $\psi_{t+k|t} \equiv \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}}$, then

$$\begin{aligned}
&\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} \left(Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) \right] \\
&= \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} \left(Y_{t+k|t} + (P_t^* - \psi_{t+k|t})(-\varepsilon) \frac{Y_{t+k|t}}{P_t^*} \right) \right] \\
&= \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} Y_{t+k|t} \left[1 + (P_t^* - \psi_{t+k|t}) \frac{-\varepsilon}{P_t^*} \right] \right] \\
&= \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[Q_{t,t+k} Y_{t+k|t} \left(1 - \varepsilon + \varepsilon \frac{\psi_{t+k|t}}{P_t^*} \right) \right] = 0.
\end{aligned}$$

Solving for the optimal price:

$$P_t^* = \mu \frac{\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t (Q_{t,t+k} Y_{t+k|t} \psi_{t+k|t})}{\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t (Q_{t,t+k} Y_{t+k|t})}, \tag{13}$$

where $\mu \equiv \frac{\varepsilon}{\varepsilon - 1}$.

In the case of flexible price, i.e., $\theta = 0$, (11) becomes $P_t^* = \mu \psi_{t|t}$. Henceforth, μ is referred to as the desired or frictionless markup.

Denote $MC_t^r = \frac{\psi_t}{P_t}$ as the real marginal cost. Consider the log-linearized form of (13)[2]:

$$\hat{p}_t^* = (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t (\hat{mc}_{t+k|t}^r + \hat{p}_{t+k}). \tag{14}$$

The forward-looking equation:

$$\hat{p}_t^* = (1 - \beta\theta)(\hat{mc}_{t|t}^r + \hat{p}_t) + \beta\theta \mathbb{E}_t \hat{p}_{t+1}^*. \tag{15}$$

[2]See Appendix 5.1 for complete derivation.

Substitute (15) into (11), we obtain

$$\begin{aligned}\hat{p}_t &= \theta\hat{p}_{t-1} + (1-\theta)\hat{p}_t^* \\ &= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)(\hat{mc}_{t|t}^r + \hat{p}_t) + (1-\theta)\beta\theta\mathbb{E}_t\hat{p}_{t+1}^* \\ &= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)(\hat{mc}_{t|t}^r + \hat{p}_t) + (1-\theta)\beta\theta\frac{\mathbb{E}_t\hat{p}_{t+1} - \theta\hat{p}_t}{1-\theta} \\ &= \theta\hat{p}_{t-1} + (1-\theta)(1-\beta\theta)(\hat{mc}_{t|t}^r + \hat{p}_t) + \beta\theta\mathbb{E}_t\hat{p}_{t+1} - \beta\theta^2\hat{p}_t.\end{aligned}$$

Subtract this equation by $(1-\theta)\hat{p}_t$ and rearrange yields

$$\begin{aligned}\theta\hat{\pi}_t &\equiv \theta(\hat{p}_t - \hat{p}_{t-1}) \\ &= (1-\theta)(1-\beta\theta)\hat{mc}_{t|t}^r + \beta\theta(\mathbb{E}_t\hat{p}_{t+1} - \hat{p}_t) \\ &= (1-\theta)(1-\beta\theta)\hat{mc}_{t|t}^r + \beta\theta\mathbb{E}_t\hat{\pi}_{t+1}.\end{aligned}$$

The forward-looking equation for inflation:

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{mc}_{t|t}^r.$$

3 Non-Policy Equilibrium

Market clearing in the goods market requires

$$Y_t(i) = C_t(i), \quad \forall i \in [0,1].$$

Letting aggregate output be defined as $Y_t \equiv \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}}$ it follows that $Y_t = C_t$, i.e., $\hat{y}_t = \hat{c}_t$. Combining with (10), we obtain the equilibrium condition of good market clearing:

$$\hat{y}_t = \mathbb{E}_t\hat{y}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}). \tag{16}$$

Market clearing in the labor market requires

$$N_t = \int_0^1 N_t(i)\mathrm{d}i.$$

Combine with the production function, we obtain

$$\begin{aligned}N_t &= \int_0^1 \left[\frac{Y_t(i)}{A_t}\right]^{\frac{1}{1-\alpha}}\mathrm{d}i \\ &= \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon}{1-\alpha}}\mathrm{d}i.\end{aligned}$$

Taking the square $1-\alpha$ yields

$$N_t^{1-\alpha} = \frac{Y_t}{A_t}\left[\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon}{1-\alpha}}\mathrm{d}i\right]^{1-\alpha}.$$

Denote $D_t = \left[\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon}{1-\alpha}} \mathrm{d}i\right]^{1-\alpha}$ measuring price (output) dispersion across firms, we have

$$Y_t = A_t N_t^{1-\alpha} D_t^{-1}.$$

The log-linearized form[3]:

$$\hat{y}_t = \hat{a}_t + (1-\alpha)\hat{n}_t. \tag{17}$$

Now we consider the average real marginal cost in the equilibrium. In firm's CMP, we have known that

$$MC_t^r = \frac{W_t}{P_t}\frac{1}{A_t(1-\alpha)N_t^{-\alpha}}.$$

The log-linearized form:

$$\begin{aligned}
\hat{mc}_t^r &= \hat{w}_t - \hat{p}_t - (\hat{a}_t - \alpha\hat{n}_t) \\
&= \hat{w}_t - \hat{p}_t - \frac{1}{1-\alpha}(\hat{a}_t - \alpha\hat{y}_t).
\end{aligned}$$

For firms that set price at t and remains it at $t+k$:

$$\begin{aligned}
\hat{mc}_{t+k|t}^r &= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \\
&= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k|t}) \\
&= \hat{w}_{t+k} - \hat{p}_{t+k} - \frac{1}{1-\alpha}(\hat{a}_{t+k} - \alpha\hat{y}_{t+k}) - \frac{\alpha}{1-\alpha}(\hat{y}_{t+k} - \hat{y}_{t+k|t}) \\
&= \hat{mc}_{t+k}^r - \frac{\alpha\varepsilon}{1-\alpha}(\hat{p}_t^* - \hat{p}_{t+k}),
\end{aligned} \tag{18}$$

where the last line uses (12).

Substitute (18) into (14), we have

$$\begin{aligned}
p_t^* &= \Gamma(\hat{mc}_{t+k|t}^r + \hat{p}_{t+k}) \\
&= \Gamma\left[\hat{mc}_{t+k}^r - \frac{\alpha\varepsilon}{1-\alpha}(\hat{p}_t^* - \hat{p}_{t+k}) + \hat{p}_{t+k}\right] \\
&= -\frac{\alpha\varepsilon}{1-\alpha}\hat{p}_t^* + \Gamma\left(\hat{mc}_{t+k}^r + \Theta^{-1}\hat{p}_{t+k}\right),
\end{aligned}$$

where $\Gamma(\cdot) \equiv (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k\mathbb{E}_t(\cdot)$, and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\varepsilon} \leq 1$.

Solving for the optimal price in terms of average real marginal cost:

$$\begin{aligned}
\hat{p}_t^* &= \Gamma(\Theta\hat{mc}_{t+k}^r + \hat{p}_{t+k}) \\
&= (1-\beta\theta)(\Theta\hat{mc}_t^r + \hat{p}_t) + \beta\theta\mathbb{E}_t\hat{p}_{t+1}^*.
\end{aligned} \tag{19}$$

Similarly, substitute (19) into (11), then subtract it by $(1-\theta)\hat{p}_t$ and arrange yields

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \lambda\hat{mc}_t^r, \tag{20}$$

[3] We can prove that $d_t = \ln D_t$ is equal to zero up to a first-order approximation in a neighborhood of $\pi = 0$, see Appendix 5.2.

where $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$.

Solving forward, we obtain

$$\hat{\pi}_t = \lambda \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \hat{mc}^r_{t+k}.$$

Re-consider the log-linearized form of average real marginal cost, using (9) and (17) yields

$$\begin{aligned} \hat{mc}^r_t &= (\hat{w}_t - \hat{p}_t) - (\hat{a}_t - \alpha \hat{n}_t) \\ &= (\sigma \hat{y}_t + \varphi \hat{n}_t) - \frac{1}{1-\alpha}(\hat{a}_t - \alpha \hat{y}_t) \\ &= \sigma \hat{y}_t + \frac{\varphi}{1-\alpha}(\hat{y}_t - \hat{a}_t) - \frac{1}{1-\alpha}(\hat{a}_t - \alpha \hat{y}_t) \\ &= \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\hat{y}_t - \frac{\varphi+1}{1-\alpha}\hat{a}_t. \end{aligned} \tag{21}$$

Under flexible prices the real marginal cost is constant ($MC^r_t = \mu^{-1}$), thus $\hat{mc}^r_t = 0$. Defining the natural level of output, denoted by y^n_t, as the equilibrium level of output under flexible prices:

$$\hat{y}^n_t = \frac{\varphi+1}{\varphi+\alpha+\sigma-\alpha\sigma}\hat{a}_t.$$

Then we have

$$\hat{mc}^r_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{y}_t, \tag{22}$$

where $\tilde{y}_t \equiv \hat{y}_t - \hat{y}^n_t$ is denoted as the output gap, the deviation between the real output and the natural level of output.

Substitute (22) into (20), we obtain the New Keynesian Philips Curve (*abbr.* NKPC):

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t, \tag{23}$$

where $\kappa \equiv \lambda\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) > 0$.

Rewrite (16) in terms of the output gap as

$$\tilde{y}_t = \mathbb{E}_t(\hat{y}^n_{t+1} - \hat{y}^n_t) + \mathbb{E}_t \tilde{y}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}).$$

Define the natural rate of interest $\hat{r}^n_t \equiv \sigma \mathbb{E}_t(\hat{y}^n_{t+1} - \hat{y}^n_t)$. Then rewriting the equation in terms of $\hat{r}^n_t$, we obtain the Dynamic IS Curve (*abbr.* DISC):

$$\tilde{y}_t = -\sigma^{-1}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}^n_t) + \mathbb{E}_t \tilde{y}_{t+1}. \tag{24}$$

Under the assumption that the effects of nominal rigidities vanish asymptotically, $\lim_{T\to\infty} \mathbb{E}_t \tilde{y}_{t+T} = 0$ In that case one can solve (24) forward to yield

$$\tilde{y}_t = -\sigma^{-1} \sum_{k=0}^{\infty} \mathbb{E}_t(\hat{r}_{t+k} - \hat{r}^n_{t+k}),$$

where $\hat{r}_t^n \equiv \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}$ is the (log-linearized) expected real return on a one period bond (i.e., the real interest rate).

In the non-policy block of basic New Keynesian model, The NKPC determines inflation given a path for the output gap, and The DISC determines output gap given a path for the (exogenous) natural rate and the actual real rate. In order to close the model, we need to supplement these two equations with one or more equations determining how the nominal interest rate $\hat{i}_t$ evolves over time, i.e., with a description of how the monetary policy is conducted.

4 Monetary Authority

Consider the equilibrium under a simple interest rate rule of the form (as known as the Taylor rule)

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + v_t, \tag{25}$$

where v_t is an exogenous (possibly stochastic) component with zero mean. Assume ϕ_π and ϕ_y are non-negative coefficients, chosen by the monetary authority.

Combine (25) with (24), the DISC becomes

$$\tilde{y}_t = -\sigma^{-1}[\phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t - \mathbb{E}_t \hat{\pi}_{t+1} - (\hat{r}_t^n - v_t)] + \mathbb{E}_t \tilde{y}_{t+1}.$$

Rewriting the NKPC and DISC:

$$\begin{aligned} (1+\sigma^{-1}\phi_y)\tilde{y}_t + \sigma^{-1}\phi_\pi \hat{\pi}_t &= \mathbb{E}_t \tilde{y}_{t+1} + \sigma^{-1}\mathbb{E}_t \hat{\pi}_{t+1} + \sigma^{-1}(\hat{r}_t^n - v_t), \\ -\kappa \tilde{y}_t + \qquad \hat{\pi}_t &= \qquad\qquad \beta \mathbb{E}_t \hat{\pi}_{t+1}. \end{aligned}$$

The matrix form becomes[4]:

$$\begin{bmatrix} \tilde{y}_y \\ \hat{\pi}_t \end{bmatrix} = \mathbf{A}_T \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \hat{\pi}_{t+1} \end{bmatrix} + \mathbf{B}_T (\hat{r}_t^n - v_t), \tag{26}$$

where $\Omega \equiv (\sigma + \phi_y + \kappa\phi_\pi)^{-1}$, $\mathbf{A}_T \equiv \Omega \begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \kappa\sigma & \kappa + \beta(\sigma+\phi_y) \end{bmatrix}$, and $\mathbf{B}_T \equiv \Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix}$.

By the Blanchard-Kahn condition, given that both the $\tilde{y}_t$ and $\hat{\pi}_t$ are non-predetermined variables, the solution to (26) is locally unique, if and only if, $\mathbf{A}_T$ has both eigenvalues within the unit circle. Under the assumption of non-negative coefficients (ϕ_π, ϕ_y) it can be shown that a necessary and sufficient condition for uniqueness is given by[5]

$$\kappa(\phi_\pi - 1) + (1-\beta)\phi_y > 0, \tag{27}$$

which is assumed to hold, unless stated otherwise.

Then we consider the monetary policy shock in the system. Assume an AR(1) process of $\{v_t\}$, that is, $v_t = \rho_v v_{t-1} + \varepsilon_t^v$, $\rho_v \in [0,1)$. A positive (negative) realization

[4] See Appendix 5.3 for complete derivation.

[5] See Appendix 5.3 too.

of ε_t^v should be interpreted as a contractionary (expansionary) monetary policy shock, leading to a rise (decline) in the nominal interest rate, given $\hat{\pi}_t$ and $\tilde{y}_t$. Because the natural rate of interest (output) is not affected by monetary shocks, $\forall t$, $\hat{r}_t^n = \hat{y}_t^n = 0$ is set for the purposes of the present exercise.

Using undetermined coefficient method, guess that[6]

$$\tilde{y}_t = \psi_{yv} v_t,$$
$$\hat{\pi}_t = \psi_{\pi v} v_t,$$

where ψ_{yv} and $\psi_{\pi v}$ are coefficients to be determined.

Verification:

$$\tilde{y}_t = -(1-\beta\rho_v)\Lambda_v v_t,$$
$$\hat{\pi}_t = -\kappa\Lambda_v v_t,$$

where $\Lambda_v \equiv [(1-\beta\rho_v)[\sigma(1-\rho_v)+\phi_y] + \kappa(\phi_\pi - \rho_v)]^{-1}$.

Since (27) is satisfied, it can be easily shown that $\Lambda_v > 0$. Then we have $-(1-\beta\rho_v)\Lambda_v < 0$, and $-\kappa\Lambda_v < 0$. Hence, If $v_t \uparrow$, then $\hat{\pi}_t \downarrow$, $\tilde{y}_t \downarrow$, and $\hat{y}_t \downarrow$.

Similarly, we can solve the form of (log-linearized) real interest $\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}$ as

$$\hat{r}_t = \sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v v_t,$$

thus we have $\sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v > 0$.

Hence,

$$\hat{i}_t = [\sigma(1-\rho_v)(1-\beta\rho_v) - \kappa\rho_v]\Lambda_v v_t.$$

If $v_t \uparrow$, then $\hat{r}_t \uparrow$. But if the persistence of the monetary policy shock ρ_v is sufficiently high, $\hat{i}_t \downarrow$, since The downward adjustment in the nominal rate induced by the decline in inflation and the output gap more than offsetting the direct effect of a higher v_t.

5 Appendix

5.1 Log-Linearization of the Optimal Price

Consider the optimal price:

$$P_t^* = \mu \frac{\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t(Q_{t,t+k} Y_{t+k|t} \psi_{t+k|t})}{\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t(Q_{t,t+k} Y_{t+k|t})}. \tag{28}$$

Detrending the the stochastic discount factor for nominal payoffs, we have

$$\tilde{Q}_{t,t+k} \equiv \beta^{-k} Q_{t,t+k} = \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}.$$

[6] See Appendix 5.4 for complete derivation.

Also we have known that

$$Y_{t+k|t} = Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}. \tag{29}$$

Rewrite (28) in terms of $\tilde{Q}_{t,t+k}$, then combine with (29), we obtain

$$P_t^* \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k} = \mu \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t \tilde{Q}_{t,t+k} P_{t+k}^{\varepsilon} Y_{t+k} \psi_{t+k|t}).$$

Expand the left hand side:

$$LHS_t = P_t^* \mathbb{E}_t (P_t^{\varepsilon} Y_t + \beta\theta \tilde{Q}_{t,t+1} P_{t+1}^{\varepsilon} Y_{t+1} + \cdots).$$

In steady state we have $\bar{P^*} = \bar{P}$ and $\tilde{Q} = 1$, thus,

$$\begin{aligned} l\hat{h}s_t &= \bar{P}^{1+\varepsilon}\bar{Y}(\hat{p}_t^* + \varepsilon\hat{p}_t + \hat{y}_t) + \bar{P}^{1+\varepsilon}\bar{Y}\beta\theta\mathbb{E}_t(\hat{p}_t^* + \varepsilon\hat{p}_{t+1} + \hat{y}_{t+1} + \hat{\tilde{q}}_{t,t+1}) + \cdots \\ &= \bar{P}^{1+\varepsilon}\bar{Y}\sum_{k=0}^{\infty}(\beta\theta)^k \hat{p}_t^* + \bar{P}^{1+\varepsilon}\bar{Y}\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t(\varepsilon\hat{p}_{t+k}\hat{y}_{t+k} + \hat{\tilde{q}}_{t,t+k}) \\ &= \bar{P}^{1+\varepsilon}\bar{Y}\frac{1}{1-\beta\theta}\hat{p}_t^* + \bar{P}^{1+\varepsilon}\bar{Y}\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t(\varepsilon\hat{p}_{t+k}\hat{y}_{t+k} + \hat{\tilde{q}}_{t,t+k}). \end{aligned} \tag{30}$$

Expand the right hand side using $\psi_t = MC_t^r P_t$:

$$RHS_t = \mu\mathbb{E}_t(\tilde{Q}_t MC_t^r P_t^{1+\varepsilon} Y_t + \beta\theta\tilde{Q}_{t,t+1} MC_{t+1|t}^r P_{t+1}^{1+\varepsilon} Y_{t+1} + \cdots).$$

In steady state we have $\bar{MC}^r = \mu^{-1}$, i.e., $\bar{P} = \mu\bar{\psi}$, thus

$$\begin{aligned} r\hat{h}s_t &= \mu\bar{MC}_t^r \bar{P}^{1+\varepsilon}\bar{Y}[\hat{\tilde{q}}_t + \hat{mc}_t^r + (1+\varepsilon)\bar{p}_t + \bar{y}_t] \\ &\quad + \mu\bar{MC}_t^r \bar{P}^{1+\varepsilon}\bar{Y}[\hat{\tilde{q}}_{t,t+1} + \hat{mc}_{t+1|t}^r + (1+\varepsilon)\bar{p}_{t+1} + \bar{y}_{t+1}] + \cdots \\ &= \bar{P}^{1+\varepsilon}\bar{Y}\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t[\hat{\tilde{q}}_{t,t+k} + \hat{mc}_{t+k|t}^r + (1+\varepsilon)\bar{p}_{t+k} + \bar{y}_{t+k}]. \end{aligned} \tag{31}$$

Subtract (31) by (30) (equal to zero) and rearrange yields

$$\hat{p}_t^* = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t(\hat{mc}_{t+k|t}^r + \hat{p}_{t+k}).$$

5.2 Price Dispersion

By the definition of the price index

$$\begin{aligned} 1 &= \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{1-\varepsilon} \mathrm{d}i \\ &= \int_0^1 \exp[(1-\varepsilon)\Delta p_t(i)]\mathrm{d}i \\ &= 1 + (1-\varepsilon)\int_0^1 \Delta p_t(i)\mathrm{d}i + \frac{(1-\varepsilon)^2}{2}\int_0^1 \Delta p_t(i)^2\mathrm{d}i + O[\Delta p_t(i)^3], \end{aligned} \tag{32}$$

where $\Delta p_t(i) \equiv p_t(i) - p_t$ is the (log) derivation of price, and $O[\Delta p_t(i)^3] = 0$ in a neighborhood of steady state.

Hence, we have

$$p_t \approx \mathbb{E}_i p_t(i) + \frac{1-\varepsilon}{2} \int_0^1 \Delta p_t(i)^2 \mathrm{d}i.$$

Thus,

$$\int_0^1 \Delta p_t(i)^2 \mathrm{d}i \approx \int_0^1 [p_t(i) - \mathbb{E}_i p_t(i)]^2 \mathrm{d}i \equiv \mathrm{Var}_i p_t(i). \tag{33}$$

In addition,

$$\begin{aligned}
\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon}{1-\alpha}} \mathrm{d}i &= \int_0^1 \exp\left[-\frac{\varepsilon}{1-\alpha}\Delta p_t(i)\right] \mathrm{d}i \\
&= 1 - \frac{\varepsilon}{1-\alpha}\int_0^1 \Delta p_t(i)\mathrm{d}i + \frac{1}{2}\left(\frac{\varepsilon}{1-\alpha}\right)^2 \int_0^1 \Delta p_t(i)^2 \mathrm{d}i + O[\Delta p_t(i)^3] \\
&= 1 + \frac{1}{2}\frac{\varepsilon(1-\varepsilon)}{1-\alpha}\int_0^1 \Delta p_t(i)^2 \mathrm{d}i + \frac{1}{2}\left(\frac{\varepsilon}{1-\alpha}\right)^2 \int_0^1 \Delta p_t(i)^2 \mathrm{d}i \\
&= 1 + \frac{1}{2}\frac{\varepsilon}{1-\alpha}\frac{1}{\Theta}\int_0^1 \Delta p_t(i)^2 \mathrm{d}i \\
&\approx 1 + \frac{1}{2}\frac{\varepsilon}{1-\alpha}\frac{1}{\Theta}\mathrm{Var}_i p_t(i),
\end{aligned}$$

where the third equality follows from (32) and the last equality follows from (33).

Using the definition of d_t,

$$d_t \equiv (1-\alpha)\ln \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\frac{\varepsilon}{1-\alpha}} \mathrm{d}i \approx \frac{1}{2}\frac{\varepsilon}{\Theta}\mathrm{Var}_i p_t(i).$$

Thus, $d_t = 0$ in a neighborhood of steady state since $\mathrm{Var}_i p_t(i) = 0$.

5.3 The System under an Interest Rate Rule

Consider the NKPC and DISC:

$$\begin{aligned}
(1+\sigma^{-1}\phi_y)\tilde{y}_t + \sigma^{-1}\phi_\pi \hat{\pi}_t &= \mathbb{E}_t \tilde{y}_{t+1} + \sigma^{-1}\mathbb{E}_t \hat{\pi}_{t+1} + \sigma^{-1}(\hat{r}_t^n - v_t), \\
-\kappa \tilde{y}_t + \qquad \hat{\pi}_t &= \qquad \beta \mathbb{E}_t \hat{\pi}_{t+1}.
\end{aligned}$$

The matrix form becomes

$$\begin{bmatrix} 1+\sigma^{-1}\phi_y & \sigma^{-1}\phi_\pi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_y \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \hat{\pi}_{t+1} \end{bmatrix} + \begin{bmatrix} \sigma^{-1} \\ 0 \end{bmatrix} (\hat{r}_t^n - v_t).$$

Since

$$\begin{bmatrix} 1+\sigma^{-1}\phi_y & \sigma^{-1}\phi_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 1+\sigma^{-1}\phi_y & \sigma^{-1}\phi_\pi \\ -\kappa & 1 \end{bmatrix}}{\begin{vmatrix} 1+\sigma^{-1}\phi_y & \sigma^{-1}\phi_\pi \\ -\kappa & 1 \end{vmatrix}} = \Omega \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa\sigma & \sigma+\phi_y \end{bmatrix},$$

where $\Omega \equiv (\sigma + \phi_y + \kappa\phi_\pi)^{-1}$, the system becomes

$$
\begin{aligned}
\begin{bmatrix} \tilde{y}_y \\ \hat{\pi}_t \end{bmatrix} &= \Omega \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa\sigma & \sigma + \phi_y \end{bmatrix} \begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \hat{\pi}_{t+1} \end{bmatrix} + \Omega \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa\sigma & \sigma + \phi_y \end{bmatrix} \begin{bmatrix} \sigma^{-1} \\ 0 \end{bmatrix} (\hat{r}_t^n - v_t) \\
&= \Omega \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \kappa\sigma & \kappa + \beta(\sigma + \phi_y) \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \hat{\pi}_{t+1} \end{bmatrix} + \Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix} (\hat{r}_t^n - v_t) \\
&\equiv \mathbf{A}_T \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \hat{\pi}_{t+1} \end{bmatrix} + \mathbf{B}_T (\hat{r}_t^n - v_t).
\end{aligned}
$$

By the Blanchard-Kahn condition, given that both the $\tilde{y}_t$ and $\hat{\pi}_t$ are nonpredetermined variables, the solution to this system is locally unique, if and only if, $\mathbf{A}_T$ has both eigenvalues within the unit circle.

Consider the characteristic polynomial of $\mathbf{A}_T$:

$$
p(\xi) = \xi^2 + \varpi_1 \xi + \varpi_0,
$$

where

$$
\begin{aligned}
\varpi_1 &= -\Omega[\sigma + \kappa + \beta(\sigma + \phi_y)] = -\frac{\kappa + \sigma + \beta\sigma + \beta\phi_y}{\sigma + \phi_y + \kappa\phi_\pi}, \\
\varpi_0 &= \Omega \begin{vmatrix} \sigma & 1 - \beta\phi_\pi \\ \kappa\sigma & \kappa + \beta(\sigma + \phi_y) \end{vmatrix} = \frac{\beta\sigma}{\sigma + \phi_y + \kappa\phi_\pi}.
\end{aligned}
$$

Both eigenvalues of $\mathbf{A}_T$ are inside the unit circle if and only if both of the following conditions hold

$$
\begin{aligned}
|\varpi_0| &< 1, \\
|\varpi_1| &< 1 + \varpi_0,
\end{aligned}
$$

which implies

$$
\phi_y + \kappa\phi_\pi + (1 - \beta)\sigma > 0, \tag{34}
$$

$$
\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y > 0. \tag{35}
$$

Since $0 < \beta < 1$, the condition (34) is trivially satisfied. Thus, (35) is what we concern. Additionally, we assume $\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y \neq 0$, which rules out eigenvalues on the unit circle.

5.4 The Undetermined Coefficient Method under a Monetary Policy Shock

The form we guessed is:

$$
\begin{aligned}
\tilde{y}_t &= \psi_{yv} v_t, \\
\hat{\pi}_t &= \psi_{\pi v} v_t,
\end{aligned}
$$

where ψ_{yv} and $\psi_{\pi v}$ are coefficients to be determined.

Rewrite the DISC

$$\tilde{y}_t = -\sigma^{-1}[\phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t - \mathbb{E}_t \hat{\pi}_{t+1} - (\hat{r}_t^n - v_t)] + \mathbb{E}_t \tilde{y}_{t+1}, \tag{36}$$

and the NKPC

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{y}_t. \tag{37}$$

Insert the guessed form into (36),

$$\psi_{yv} v_t = -\sigma^{-1}[\phi_\pi \psi_{\pi v} v_t + \phi_y \psi_{yv} v_t + v_t - \rho_v \psi_{\pi v} v_t] + \rho_v \psi_{yv} v_t,$$

which implies

$$[\sigma(1-\rho_v) + \phi_y]\psi_{yv} = (\rho_v - \phi_\pi)\psi_{\pi v} - 1. \tag{38}$$

Turning to (37),

$$\psi_{\pi v} v_t = \beta \rho_v \psi_{\pi v} v_t + \kappa \psi_{yv} v_t,$$

which implies

$$\psi_{yv} = \frac{1-\beta\rho_v}{\kappa}\psi_{\pi v}. \tag{39}$$

Insert (39) into (38),

$$\frac{1-\beta\rho_v}{\kappa}\psi_{\pi v} = (\rho_v - \phi_\pi)\psi_{\pi v} - 1,$$

and solve for $\psi_{\pi v}$ as

$$\psi_{\pi v} = -\kappa \Lambda_v,$$

where $\Lambda_v \equiv [(1-\beta\rho_v)[\sigma(1-\rho_v)+\phi_y] + \kappa(\phi_\pi - \rho_v)]^{-1}$.

Then the solution becomes

$$\begin{aligned}
\tilde{y}_t &= -(1-\beta\rho_v)\Lambda_v v_t, \\
\hat{\pi}_t &= -\kappa \Lambda_v v_t.
\end{aligned}$$

Consider the real interest,

$$\begin{aligned}
\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} \\
&= \phi_\pi \psi_{\pi v} v_t + \phi_y \psi_{yv} v_t + v_t + \kappa \Lambda_v \rho_v v_t \\
&= [-\Lambda_v[\kappa(\phi_\pi - \rho_v) + (1-\beta\rho_v)\phi_y] + 1]v_t \\
&= [-\Lambda_v[\Lambda_v^{-1} - \sigma(1-\rho_v)(1-\beta\rho_v)] + 1]v_t \\
&= \sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v v_t,
\end{aligned}$$

then the nominal interest becomes

$$\begin{aligned}
\hat{i}_t &= \sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v v_t - \kappa \Lambda_v \rho_v v_t \\
&= [\sigma(1-\rho_v)(1-\beta\rho_v) - \kappa\rho_v]\Lambda_v v_t.
\end{aligned}$$