Reading Notes of Straub & Werning (2020) "Positive Long-Run Capital Taxation" Part 2. Representative-Agent Model with Public Debt

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1 Environment and Previous Results

1.1 Preferences

We write the representative agent's utility as $\mathcal{V}(U_0, U_1, \dots)$ with per-period utility $U_t = U(c_t, n_t)$ depending on consumption c_t and labor supply n_t. Assume that $\mathcal{V}$ is increasing in every argument and satisfies the recursion below:

$$V_t = W(U_t, V_{t+1}) = \mathcal{V}(U_t, U_{t+1}, \dots),$$

where $W(U, V')$ is an aggregator function.

Assume that both $U(c, n)$ and $W(U, V')$ are twice continuously differentiable, with W_U, W_V, $U_c > 0$ and $U_n < 0$. Consumption and leisure are taken to be normal goods, that is,

$$\frac{U_{cc}}{U_c} - \frac{U_{nc}}{U_n} \leq 0 \quad \text{and} \quad \frac{U_{cn}}{U_c} - \frac{U_{nn}}{U_n} \leq 0,$$

with at least one strict inequality.

Define $\bar{U}(V)$ as the solution to $V = W(\bar{U}(V), V)$ (as a fixed point) and let $\bar{\beta}(V) := W_V(\bar{U}(V), V)$ denote the steady-state discount factor[1]. By equation (1.1 RC), we have $\mathrm{d}\mathcal{V} = W_U \mathrm{d}U + W_V + \mathrm{d}\mathcal{V}$. According to the strict monotonicity of $\mathcal{V}$, we obtain that

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}U} = \frac{W_U}{1 - W_V} > 0 \Rightarrow \bar{\beta}(V) \in (0, 1),$$

since $W_U > 0$ and $W_V > 0$.

1.2 Technology

Output is obtained from capital and labor using a neoclassical CRS production function $F(k_t, n_t)$ satisfying standard conditions.

The resource constraint in period t is then

$$c_t + k_{t+1} + g_t \leq F(k_t, n_t) + (1 - \delta)k_t, \tag{1.1 RC}$$

where $\delta \in [0, 1]$ denotes the depreciate rate of capital, and the initial capital $k_0 > 0$ and the sequence for government consumption $\{g_t\}$ is given exogenously.

[1] The additively separable utility case amounts to the particular linear choice $W(U, V') = U + \beta V'$, with a constant $\beta \in (0, 1)$. But nonlinear aggregators allow local discounting to vary with U and V', which ensures that zero-tax results are not driven by an infinite long-run elasticity of savings, as we discussed in Part 1.

1.3 Markets and Taxes

Labor and capital markets are perfectly competitive, yielding before-tax wages and rates of return given by $w_t^* = F_n(k_t, n_t)$ and $R_t^* = F_k(k_t, n_t) + 1 - \delta$. The after-tax wage and return relate to their before-tax counterparts by $w_t = (1 - \tau_t^n) w_t^*$ and $R_t = (1 - \tau_t)(R_t^* - 1) + 1$. Following the original model, we allow for an indirect constraint on the capital tax rate given by $R_t \geq 1$. For positive before-tax interest rates $R_t^* - 1$ this is precisely equivalent to assuming $\tau_t \leq 1$.

1.4 Agent's Behavior

The agent's UMP:

$$\max_{\{V_t, c_t, n_t, R_0\}} \mathcal{V}(U_0, U_1, \dots)$$

s.t.

$$\begin{aligned} c_0 + a_1 &\leq w_0 n_0 + R_0 k_0 + R_0^b b_0, \\ c_t + a_{t+1} &\leq w_t n_t + R_t a_t, \quad t = 1, 2, \dots, \\ \lim_{t \to \infty} \frac{a_{t+1}}{\prod_{s=1}^{t} R_s} &= 0, \end{aligned}$$

with w_t and R_t given.

Total assets $a_t = k_t + b_t$ are composed of capital k_t and government debt b_t. With perfect foresight, the non-arbitrage condition implies the same return in equilibrium $\forall t \geq 1$. This is not true for the initial period, where we allow possibly different returns on capital and debt.

1.5 Planning Problem

Firstly we derive for the implementability condition in this economy. In agent's UMP, it is standard to write

$$\frac{\mathcal{V}_{n,t}}{\mathcal{V}_{c,t}} = -w_t,$$

and

$$\mathcal{V}_{c,t} = \mathcal{V}_{c,t+1} R_{t+1}.$$

Let $q_t \equiv 1/\prod_{s=1}^{t} R_s$, $t = 1, 2, \dots$, with $q_0 \equiv 1$. Then $R_{t+1} = q_t / q_{t+1}$, and $q_t = \mathcal{V}_{c,t}/\mathcal{V}_{c,0}$. Combining the two budget constraints as

$$\sum_{t=0}^{\infty} q_t (c_t - w_t n_t) = R_0 k_0 + R_0^b b_0.$$

Since

$$\text{LHS} = \sum_{t=0}^{\infty} q_t (c_t - w_t n_t) = \frac{1}{\mathcal{V}_{c,0}} \sum_{t=0}^{\infty} (\mathcal{V}_{c,t} c_t - \mathcal{V}_{c,t} w_t n_t) = \frac{1}{\mathcal{V}_{c,0}} \sum_{t=0}^{\infty} (\mathcal{V}_{c,t} c_t + \mathcal{V}_{n,t} n_t),$$

we obtain the implementability condition (as well as the IC constraint)

$$\sum_{t=0}^{\infty} (\mathcal{V}_{c,t} c_t + \mathcal{V}_{n,t} n_t) = \mathcal{V}_{c,0} (R_0 k_0 + R_0^b b_0). \tag{1.2 ICC}$$

To enforce the constraints on the taxation of capital in periods $t = 1, 2, \dots$, we impose

$$\mathcal{V}_{c,t} = R_{t+1} \mathcal{V}_{c,t+1}, \tag{1.3}$$

$$R_t \geq 1. \tag{1.4}$$

Here, the first equation is literally the agent's Euler equation.

The planning problem can be described as

$$\max_{\{V_t, c_t, n_t, R_0\}} \mathcal{V}(U_0, U_1, \dots)$$

s.t. equation (1.1 RC), (1.2 ICC), (1.3), and (1.4). In addition, we take R_0^b as given.

Note that the constraint (1.4) may or may not bind forever. In the next two subsection, we are interested in situations where the constraint does not bind asymptotically, i.e., it is slack after some date $T < \infty$.

1.6 Converged Multiplier: Zero-Taxation Result

Chamley (1986) provided the following result, slightly adjusted here to make explicit the need for the steady state to be interior, for multipliers to converge, and for the bounds on taxation to be asymptotically slack.

Theorem 1.1: Chamley (1986)

Suppose the optimum converges to an interior steady state where the constraints on capital taxation are asymptotically slack. Let $\tilde{\Lambda}_t = \mathcal{V}_{c,t}\Lambda_t$ denote the multiplier on the resource constraint (1.1 RC) in period t. Suppose further that the multiplier Λ_t converges to an interior point $\Lambda_t \to \Lambda > 0$. Then the tax on capital converges to zero, i.e., $R_t/R_t^* = 1$.

Proof. Consider a sufficiently late period $t > T$, so that the bounds on the capital tax rate are no longer binding. Then the FOC for k_{t+1} is simply $\tilde{\Lambda}_t = \tilde{\Lambda}_{t+1}R_{t+1}^*$, similarly to the form in the derivation of implementability condition. Using the definition of $\tilde{\Lambda}_t$ one yields

$$\mathcal{V}_{c,t}\Lambda_t = \mathcal{V}_{c,t+1}\Lambda_{t+1}R_{t+1}^*.$$

Combining with equation (1.4), since $\Lambda_t \to \Lambda$, we have

$$\lim_{t\to\infty} \frac{R_{t+1}}{R_{t+1}^*} = \lim_{t\to\infty} \frac{\mathcal{V}_{c,t}/\mathcal{V}_{c,t+1}}{\mathcal{V}_{c,t}\Lambda_t/\mathcal{V}_{c,t+1}\Lambda_{t+1}} = 1.$$

$\square$

1.7 General Conclusion: First-Best or Zero Wealth

In this subsection, we prove a proposition with weaker assumptions than Theorem 1.1. Specifically, we make no assumptions on multipliers and prove that the steady-state tax rate is zero[2].

Proposition 1.2

Suppose the optimal allocation converges to an interior steady state and assume the bounds on capital tax rates are asymptotically slack. Then the tax on capital is asymptotically zero. In addition, if the discount factor is locally nonconstant at the steady state, so that $\bar{\beta}'(V) \neq 0$, then either

(i) private wealth converges to zero, $a_t \to 0$; or

(ii) the allocation converges to the first-best, with a zero tax rate on labor.

[2] As long as the multiplier Λ_t converges, one does not even need to assume the allocation converges to arrive at the zero-tax conclusion, see the proof of Theorem 1.1.

Proof. In this proof, we denote by $X_{z,t}$ the derivative of quantity X w.r.t. z, evaluated at time t. To save on notation, we define $f(k,n) \equiv F(k,n) + (1-\delta)k$.

First, we exploit the recursiveness of $\mathcal{V}$ to recast the IC constraint (1.2 ICC) entirely in terms of V_t and $W(U, V')$. Let $\beta_t \equiv \prod_{s=0}^{t-1} W_{V,s}$. By the definition of the aggregator, we can derive

$$\begin{aligned}
\mathcal{V}_{c,t} &= \frac{\partial}{\partial c_t} W\Big(U_t, W\big(U_{t+1}, W(U_{t+2}, \dots)\big)\Big) \\
&= \frac{\partial W}{\partial U_t}\frac{\partial U_t}{\partial c_t} \cdot \prod_{s=0}^{t-1} \frac{\partial W}{\partial V_s} \\
&= \beta_t W_{U,t} U_{c,t}.
\end{aligned}$$

Similarly, we have $\mathcal{V}_{n,t} = \beta_t W_{U,t} U_{n,t}$. Thus the IC constraint (1.2 ICC) can be written as

$$\sum_{t=0}^{\infty} \beta_t W_{U,t}(U_{c,t}c_t + U_{n,t}n_t) = W_{U,0}U_{c,0}(R_0 k_0 + R_0^b b_0), \tag{1.5}$$

and the planning problem becomes

$$\max_{\{V_t, c_t, n_t, R_0\}} V_0$$

s.t.

$$V_t = W(U(c_t, n_t), V_{t+1}),$$

and (1.1 RC), (1.2 ICC), (1.4).

To state the FOCs, define for each $t \geq 0$,

$$\begin{aligned}
A_{t+1} &\equiv \frac{1}{\beta_{t+1}} \frac{\partial}{\partial V_{t+1}} \sum_{s=0}^{\infty} \beta_s W_{U,s}(U_{c,s}c_s + U_{n,s}n_s), \\
B_{t+1} &\equiv \frac{1}{\beta_t} \sum_{s=0}^{\infty} \frac{\partial(\beta_s W_{U,s})}{\partial U_t}(U_{c,s}c_s + U_{n,s}n_s).
\end{aligned}$$

Assume the capital tax bound is not binding eventually, say from period T onwards. All the FOCs we derived are evaluated at time $t \geq T$.

Define the Lagrangian:

$$\begin{aligned}
\mathcal{L} = V_0 &+ \sum_{t=0}^{\infty} \beta_t v_t [V_t - W(U(c_t, n_t), V_{t+1})] + \sum_{t=0}^{\infty} \beta_t \lambda_t [f(k_t, n_t) - c_t - k_{t+1} - g_t] \\
&+ \mu \left[\sum_{t=0}^{\infty} \beta_t W_{U,t}(U_{c,t}c_t + U_{n,t}n_t) - W_{U,0}U_{c,0}(R_0 k_0 + R_0^b b_0) \right].
\end{aligned}$$

The derivative w.r.t. V_{t+1}:

$$\beta_{t+1} v_{t+1} - \beta_t v_t W_{V,t+1} + \mu \frac{\partial}{\partial V_{t+1}} \sum_{s=0}^{\infty} \beta_s W_{U,s}(U_{c,s}c_s + U_{n,s}, n_s) = 0,$$

that is,

$$-v_t + v_{t+1} + \mu A_{t+1} = 0. \tag{1.6}$$

The derivative w.r.t. c_t:

$$-\beta_t v_t W_{U,t} U_{c,t} - \beta_t \lambda_t + \mu \beta_t W_{U,t}(U_{c,t} + U_{cc,t}c_t + U_{nc,t}n_t) + \mu \sum_{s=0}^{\infty} \frac{\partial \Gamma_s}{\partial U_t} U_{c,t} = 0,$$

where $\Gamma_s \equiv \beta_s W_{U,s}(U_{c,s}c_s + U_{n,s}n_s)$. Then we obtain

$$-v_t W_{U,t}U_{c,t} + \mu W_{U,t}(U_{c,t} + U_{cc,t}c_t + U_{nc,t}n_t) + \mu B_t U_{c,t} = \lambda_t. \tag{1.7}$$

Similarly, we can derive the derivative w.r.t. n_t:

$$v_t W_{U,t}U_{n,t} - \mu W_{U,t}(U_{n,t} + U_{cn,t}c_t + U_{nn,t}n_t) - \mu B_t U_{n,t} = \lambda_t f_{n,t}. \tag{1.8}$$

The derivative w.r.t. k_{t+1}:

$$\beta_{t+1}\lambda_{t+1}f_{k,t+1} - \beta_t\lambda_t = 0,$$

that is,

$$-\lambda_t + \lambda_{t+1}W_{V,t}f_{k,t+1} = 0. \tag{1.9}$$

Suppose the allocation converges to an interior steady state in c, k, and n. Then U_t and V_t converge, as well as their first and second derivatives. Then we consider the convergence of agent's assets a_t. Rewriting (1.5) at time $t+1$,

$$\frac{1}{\beta_{t+1}}\sum_{s=t+1}^{\infty}\Gamma_s = W_{U,t+1}U_{c,t+1}R_{t+1}a_{t+1},$$

where the definition of Γ_s is same as above. Then taking the limit of a_{t+1},

$$\lim_{t\to\infty} a_{t+1} = \lim_{t\to\infty}\frac{1}{W_{U,t+1}U_{c,t+1}\beta_{t+1}R_{t+1}}\sum_{s=t+1}^{\infty}\Gamma_s = \frac{1}{(1-\beta)U_cR}(U_cc + U_nn),$$

where $\beta \equiv \bar{\beta}(V) = W_V \in (0,1)$. Thus,

$$a_t \to a = \frac{U_cc + U_nn}{(1-\beta)U_cR}.$$

We move to consider the convergence of two auxiliary function A_t and B_t.

Using this expression, we can rewrite A_{t+1} as

$$\begin{aligned} A_{t+1} &= \frac{W_{UV,t}}{W_{V,t}}(U_{c,t}c_t + U_{n,t}n_t) + \frac{W_{VV,t}}{W_{V,t}}\frac{1}{\beta_{t+1}}\sum_{s=t+1}^{\infty}\Gamma_s \\ &= \frac{W_{UV,t}}{W_{V,t}}(U_{c,t}c_t + U_{n,t}n_t) + W_{U,t+1}U_{c,t+1}R_{t+1}a_{t+1}, \end{aligned}$$

converges as well, to some limit A,

$$\begin{aligned} A &= \frac{\beta_U}{\beta}(U_cc + U_nn) + \frac{\beta_V}{\beta}W_UU_cRa \\ &= \frac{\beta_U}{\beta}(1-\beta)U_cRa\frac{\beta_V}{\beta}W_UU_cRa \\ &\overset{\bar{\beta}'(V)=\frac{1-W_V}{W_U}\beta_U+\beta_V}{=\!=\!=\!=\!=\!=\!=} \frac{\bar{\beta}'(V)}{\beta}W_UU_cRa, \end{aligned}$$

where we defined $\beta_X \equiv W_{VX}$ and $X = U, V$. Similarly, we can show that B_t converges to some $B < \infty$.

Taking the limits of quantities in the FOCs (1.6), (1.7), (1.8), and (1.9), we thus find a system of equations for multipliers v_t, μ, and λ_t,

$$-v_t + v_{t+1} + \mu A = 0 \tag{1.10}$$

$$-v_t + \mu\left(1 + \frac{U_{cc}c}{U_c} + \frac{U_{nc}n}{U_c}\right) + \beta\frac{B}{W_U} = \lambda_t\frac{1}{W_UU_c} \tag{1.11}$$

$$-v_t + \mu\left(1 + \frac{U_{cn}c}{U_n} + \frac{U_{nn}n}{U_n}\right) + \beta\frac{B}{W_U} = -\lambda_t\frac{f_n}{W_UU_n} \tag{1.12}$$

$$-\lambda_t + \lambda_{t+1}\beta f_k = 0. \tag{1.13}$$

To derive our main result, we can rearrange the form of βf_k[3].
Combining (1.10) with (1.11) one yields

$$\lambda_t \frac{1}{W_U U_c} - \lambda_{t+1} \frac{1}{W_U U_c} + \mu A = 0,$$

Then rewriting equation (1.13) to substitute out λ_t,

$$\beta f_k - 1 = \frac{\lambda_t - \lambda_{t+1}}{\lambda_{t+1}} = -\frac{W_U U_c}{\lambda_{t+1}} \mu A. \tag{1.14}$$

Now we can show that the steady state capital taxes are still zero.

Lemma 1.3

At an interior steady state, capital taxes are zero, i.e., $\beta f_k = 1$.

Proof. If $A = 0$ or $\mu = 0$ the result is immediate from (1.14). In this case, we have $\lambda_t \to 0$ by (1.15). So the optimal allocation is first best to begin with.

Suppose instead that $A \neq 0$ and $\mu \neq 0$. Then, (1.11) implies that v_t and hence λ_t diverges to $\pm\infty$. Then again, $\beta f_k = 1$ follows from (1.14). □

Finally, we show that steady state labor taxes are also zero, unless $\bar{\beta}'(V) = 0$, when preferences are locally additive separable, or the steady state assets are zero $a = 0$.

Lemma 1.4

At an interior steady state, labor taxes are zero, i.e., $\tau^n \equiv 1 + \frac{U_n}{U_c f_n} = 0$ if $\bar{\beta}'(V) \neq 0$ and $a > 0$.

Proof. Combining (1.11) and (1.12) one yields

$$\mu \left(\frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} \right) - \lambda_t \frac{1}{W_U U_c} = \mu \left(\frac{U_{cn} c}{U_n} + \frac{U_{nn} n}{U_n} \right) + \lambda_t \frac{f_n}{W_U U_n},$$

that is,

$$\lambda_t \left(\frac{U_n + U_c f_n}{W_U U_c U_n} \right) = \mu_t \left(\frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} - \frac{U_{cn} c}{U_n} - \frac{U_{nn} n}{U_n} \right).$$

Thus, we can find an expression for τ^n,

$$\lambda_t \tau^n = \mu \frac{W_U U_n}{f_n} \left(\frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} - \frac{U_{cn} c}{U_n} - \frac{U_{nn} n}{U_n} \right). \tag{1.15}$$

By normality of consumption and labor, the term in brackets is negative. It is immediate from (1.15) that $\tau^n = 0$ if λ_t diverges to $\pm\infty$[4].

Suppose $\lambda_t \to \lambda \in \mathbb{R}$. If $\mu = 0$, the economy was first best to start with, thus the labor tax must be zero at any date. If $\mu \neq 0$, convergence of λ_t (or v_t) necessitates that $A = 0$ by (1.10). But by the expression of A, this is towards a contradiction, since $\bar{\beta}'(V) \neq 0$ and $a > 0$. □

□

[3]Note that $\beta f_k = R^*/R$ at steady state.

[4]Since $A \neq 0$ and μ is constant over time, v_t and thus also λ_t have a well-defined limit in $[-\infty, +\infty]$.

1.8 A Continuous-Time Model

For present purposes, and following existing literature, it is convenient to work with a continuous-time version of the model and restrict attention to additively separable preferences,

$$\int_0^\infty \mathrm{e}^{-\rho t} U(c_t, n_t)\mathrm{d}t, \quad U(c,n) = u(c) - v(n), \tag{1.16}$$

with

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$
$$v(n) = \frac{n^{1+\zeta}}{1+\zeta},$$

where $\sigma, \zeta > 0$.

The resource constraint is

$$c_t + \dot{k}_t + g = f(k_t, n_t) - \delta k_t, \tag{1.17}$$

where f has CRS with $f(0,n) = f(k,0) = 0$, is differentiable and strictly concave in each argument, and satisfies the usual Inada conditions.

Taking government consumption to be constant at $g > 0$. Denote the before-tax net interest rate by $r_t^* = f_k(k_t, n_t) - \delta$. The implementability condition is now

$$\int_0^\infty \mathrm{e}^{-\rho t}[u'(c_t)c_t - v'(n_t)n_t]\mathrm{d}t = u'(c_0)a_0, \tag{1.18}$$

where $a_0 = k_0 + b_0$ denotes initial priveate wealth, consisting of capital k_0 and government bonds b_0.

To enforce bounds on capital taxation we impose

$$\dot{\theta}_t = \theta_t(\rho - r_t), \tag{1.19}$$

and

$$r_t \geq 0, \tag{1.20}$$

where $\theta_t = u'(c_t)$ denotes the marginal utility of consumption, and r_t denotes the after-tax interest rate. Whenever $r_t^* > 0$, constraint (1.20) corresponds to a capital tax constraint $\tau_t = 1 - r_t/r_t^* \leq \bar{\tau} \equiv 1$.

The planning problem maximizes (1.16) s.t. (1.17), (1.18), (1.19), and (1.20).

2 Revisiting Chamley (1986)

Chamley (1986) formulated the following claim regarding the path for capital tax rates, which is called the bang-bang property,

Claim

There exist a time T with the following properties:

(i) For $t < T$, the constraint (1.20) is binding, that is, $r_t = 0$ and $\tau_t = 1$;

(ii) For $t > T$, $r_t = r_t^*$ and $\tau_t = 0$;

(iii) $T < \infty$.

Given $\{g_t\}$, the tax burden the government must impose varies with initial government debt b_0. As with a regular Laffer curve, there exists a maximum burden of taxes agents can finance, here we denote it by $\bar{b}$. When $b_0 > \bar{b}$, no feasible allocation exists, while there are always feasible allocations if $b_0 < \bar{b}$.

The next result shows that part (iii) of the above claim is incorrect.

Proposition 2.1

Suppose preferences are given by (1.16). Fix any initial capital stock $k_0 > 0$ and assume initial private wealth $k_0 + b_0$ is positive. Then, the bang-bang property holds, so that at any optimum of the planning problem, $\exists T \in [0, \infty]$, s.t. $\forall t < T$, $\tau_t = \bar{\tau} = 1$ and $\forall t > T$, $\tau_t = 0$. Whenever the economy is not at its first-best, T is strictly positive. Moreover:

(i) For $\sigma > 1$, $\exists \underline{b} < \bar{b}$, s.t.:

- If $b_0 = \bar{b}$, the unique optimum has $T = \infty$, and there is no feasible allocation with $T < \infty$.
- If $b_0 \in [\underline{b}, \bar{b})$, the unique optimum has $T = \infty$, but there exist feasible allocations with $T < \infty$.
- If $b_0 < \underline{b}$, any optimum has $T < \infty$.

(ii) For $\sigma = 1$:

- If $b_0 = \bar{b}$, the unique optimum has $T = \infty$, and there is no feasible allocation with $T < \infty$.
- If $b_0 < \bar{b}$, any optimum has $T < \infty$.

(iii) For $\sigma < 1$: Any optimum has $T < \infty$.

Proof. See the remaining subsections below. □

2.1 Planning Problem and FOCs

As in the statement of the proposition, we fix some positive initial level of capital $k_0 > 0$. The problem under scrutiny is

$$V(b_0) \equiv \max_{\{c_t, n_t, k_t, r_t\}} \int_0^\infty \mathrm{e}^{-\rho t}[u(c_t) - v(n_t)]\mathrm{d}t \tag{2.1}$$

s.t.

$$\dot{c}_t = c_t \frac{1}{\sigma}(r_t - \rho), \tag{2.2}$$

$$c_t + g + \dot{k}_t = f(k_t, n_t) - \delta k_t, \tag{2.3}$$

$$\int_0^\infty \mathrm{e}^{-\rho t}[u'(c_t)c_t - v'(n_t)n_t]\mathrm{d}t \geq u'(c_0)(k_0 + b_0), \tag{2.4}$$

$$c_t > 0, n_t \geq 0, k_t \geq 0, r_t \geq 0.$$

Let μ be the multiplier on the IC constraint, then this item in Hamiltonian is written as

$$\mu \left[\int_0^\infty \mathrm{e}^{-\rho t}[u'(c_t)c_t - v'(n_t)n_t]\mathrm{d}t - u'(c_0)(k_0 + b_0) \right] = \int_0^\infty \mathrm{e}^{-\rho t}\mu[u'(c_t)c_t - v'(n_t)n_t]\mathrm{d}t - u'(c_0)(k_0 + b_0)$$

Combining with the objective function, then we get the new item,

$$u(c) - v(n) + \mu[u'(c)c - v'(n)n] = [1 + \mu(1 - \sigma)]u(c) - [1 + \mu(1 + \zeta)]v(n),$$

since $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $v(n) = \frac{n^{1+\zeta}}{1+\zeta}$, $\zeta > 0$.

Then, the current-value Hamiltonian with subsidiary condition (2.4) can be written as

$$\mathcal{H} = \Phi_u u(c) - \Phi_v v(n) + \eta c \frac{1}{\sigma}(r - \rho) + \lambda[f(k, n) - \delta k - c - g], \tag{2.5}$$

where we defined $\Phi_v \equiv 1 + \mu(1+\zeta)$ and $\Phi_u \equiv 1 + \mu(1-\sigma)$; and where we denoted the costates of c_t and k_t by η_t and λ_t, respectively.

The derivative w.r.t. n_t:

$$\Phi_v v'(n_t) - \lambda_t f_n(k_t, n_t) = 0. \tag{2.6}$$

The derivative w.r.t. r_t:

$$\eta_t c_t \frac{1}{\sigma} r_t = 0,$$

or

$$r_t \begin{cases} = 0 & \text{if } \eta_t < 0 \\ \in [0, \infty) & \text{if } \eta_t = 0 \end{cases}. \tag{2.7}$$

The derivative w.r.t. c_t (or η_t):

$$\Phi_u u'(c_t) + \eta_t \frac{1}{\sigma} r_t - \eta_t \frac{1}{\sigma}\rho - \lambda_t = \rho \eta_t - \dot{\eta}_t.$$

By equation (2.7), we obtain

$$\dot{\eta}_t - \rho\eta_t = \eta_t \frac{\rho}{\sigma} + \lambda_t - \Phi_u u'(c_t). \tag{2.8}$$

The derivative w.r.t. k_t (or λ_t):

$$\lambda_t[f_k(k_t, n_t) - \delta] = \rho\lambda_t - \dot{\lambda}_t,$$

By the definition of r_t^*, we have

$$\dot{\lambda}_t = (\rho - r_t^*)\lambda_t. \tag{2.9}$$

The optimality condition for the initial state of consumption c_0 implies

$$\eta_0 = -\mu u'(c_0)(a_0 + b_0) = -\mu\sigma c_0^{-\sigma-1}(k_0 + b_0). \tag{2.10}$$

The conditions (2.5)-(2.10), together with the constraints (2.2)-(2.4) and the two TVCs

$$\lim_{t\to 0} \mathrm{e}^{-\rho t}\lambda_t k_t = 0, \tag{2.11}$$

$$\lim_{t\to 0} \mathrm{e}^{-\rho t}\eta_t c_t = 0, \tag{2.12}$$

are sufficient for an optimum if we are able to establish that the planning problem (2.1) is a concave maximization problem, or can be transformed into one using variable transformations.

The FOCs are necessary at an optimum since interiority is ensured by the imposition of Inada conditions; that is, with exception when that optimum is also maximizing the subsidiary constraint (2.4). Moreover, the above FOCs are not necessary when the optimum to (2.1) achieves the supremum in

$$\bar{b} \equiv \sup_{\{c_t, n_t, k_t\}} \frac{1}{u'(c_0)} \int_0^\infty \mathrm{e}^{-\rho t}[u'(c_t)c_t - v'(n_t)n_t]\mathrm{d}t - k_0, \tag{2.13}$$

subject to the two other constraints (2.2) and (2.3). Note that $\bar{b} \in [-\infty, +\infty]$, allowing for $\bar{b} = -\infty$ if no feasible allocation exists at all, and $\bar{b} = \infty$ if there exists a feasible allocation for any value of b_0.

Since in the case that $b_0 = \bar{b}$ the supremum in (2.13) is attained, there are still necessary FOCs the allocation satisfies, namely the ones corresponding to (2.13).

3 Revisiting Judd (1999)

3.1 Bounded Multipliers and Zero Average Capital Taxes

In this section, we consider the same planning problem to the last section. Let $\hat{\Lambda}_t = \theta_t \Lambda_t$ denote the costate for capital, that is, the current value multiplier on equation (2.3), satisfying $\dot{\hat{\Lambda}}_t = \rho \hat{\Lambda}_t - r_t^* \hat{\Lambda}_t$. Using that

$$\frac{\dot{\hat{\Lambda}}_t}{\hat{\Lambda}_t} = \frac{\dot{\theta}_t}{\theta_t} + \frac{\dot{\Lambda}_t}{\Lambda_t},$$

and (1.19), we obtain

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = r_t - r_t^*.$$

If Λ_t converges then $r_t - r_t^* \to 0$. Thus, the Chamley (1986) steady-state result actually follows by postulating the convergence of Λ_t, without assuming convergence of the allocation. Judd (1999) goes down this route, but assumes that the endogenous multiplier Λ_t remains in a bounded interval, instead of assuming that it converges.

Theorem 3.1: Judd (1999)

Let $\theta_t \Lambda_t$ denote the (current value) costate for capital in equation (2.3) and assume

$$\Lambda_t \in [\underline{\Lambda}, \overline{\Lambda}],$$

for $0 < \underline{\Lambda} < \overline{\Lambda} < \infty$. Then the cumulative distortion up to t is bounded,

$$\ln \Lambda_0 - \ln \overline{\Lambda} \leq \int_0^t (r_s - r_s^*) \mathrm{d}s \leq \ln \Lambda_0 - \ln \underline{\Lambda},$$

and the average distortion converges to zero,

$$\frac{1}{t} \int_0^t (r_s - r_s^*) \mathrm{d}s \to 0.$$

Proof. Since

$$\ln \Lambda_t - \ln \Lambda_0 = \int_0^t \frac{\dot{\Lambda}_s}{\Lambda_s} \mathrm{d}s = \int_0^t (r_s - r_s^*) \mathrm{d}s$$

and

$$\ln \underline{\Lambda} \leq \ln \Lambda_t \leq \ln \overline{\Lambda},$$

we obtain that

$$\ln \Lambda_0 - \ln \overline{\Lambda} \leq \int_0^t (r_s - r_s^*) \mathrm{d}s \leq \ln \Lambda_0 - \ln \underline{\Lambda}.$$

Dividing by $t > 0$ one yields

$$\frac{\ln \Lambda_0 - \ln \overline{\Lambda}}{t} \leq \frac{1}{t} \int_0^t (r_s - r_s^*) \mathrm{d}s \leq \frac{\ln \Lambda_0 - \ln \underline{\Lambda}}{t}.$$

Let $t \to \infty$, by Squeeze Theorem we have

$$\frac{1}{t} \int_0^t (r_s - r_s^*) \mathrm{d}s \to 0.$$

□

In this proof, we did not invoke FOCs for r_t nor for τ_t. This leads to a question: do the bounds on Λ_t essentially assume the result? We next show that, there is no guarantee that endogenous object Λ_t remains bounded away from zero, making Theorem 3.1 inapplicable.

Proposition 3.2

At the optimum described in Proposition 2.1, we have that $\Lambda_t \to 0$ as $t \to \infty$. Thus, in this case the assumption on the endogenous multiplier Λ_t is violated.

Proof. The multiplier $\mathrm{e}^{-\rho t}\hat{\Lambda}_t$ represents the planner's social marginal value of resource at time t. Thus,

$$\begin{aligned}
\mathrm{MRS}^{\text{social}}_{t,t+s} &= \mathrm{e}^{-\rho s}\frac{\hat{\Lambda}_{t+s}}{\hat{\Lambda}_t} \\
&= \exp[-\rho s + \ln \hat{\Lambda}_{t+s} - \ln \hat{\Lambda}_t] \\
&= \exp\left[-\rho s + \int_t^{t+s} (\rho - r^*_{\bar{s}}\mathrm{d}\bar{s})\right] \\
&= \exp \int_0^t r^*_{t+\bar{s}}\mathrm{d}\bar{s},
\end{aligned}$$

which is equated to the MRT given the assumption $\bar{\tau} = 1$.

Similarly, the private agent's MRS is

$$\mathrm{MRS}^{\text{private}}_{t,t+s} = \mathrm{e}^{-\rho s}\frac{\theta_{t+s}}{\theta_t} = \exp \int_0^t r_{t+\bar{s}}\mathrm{d}\bar{s},$$

where θ_t represents marginal utility.

It follows that

$$\frac{\Lambda_{t+s}}{\Lambda_t} = \frac{\mathrm{MRS}^{\text{social}}_{t,t+s}}{\mathrm{MRS}^{\text{private}}_{t,t+s}}.$$

Thus, the wedge $\Lambda_{t+s}/\Lambda_t = \exp \int_0^s (r_{t+\bar{s}} - r^*_{t+\bar{s}})\mathrm{d}\bar{s}$ is the only source of nonzero taxes. Whenever Λ_t is constant, social and private MRS values coincide and the intertemporal wedge is zero, $r_t = r^*_t$. If Λ_t is enclosed in a bounded interval, the same conclusion holds on average. □

3.2 Exploding Consumption Taxes

Judd (1999) also offers an intuitive interpretation for the zero-tax result based on the observation that an indefinite tax on capital is equivalent to an ever-increasing tax on consumption. If the marginal cost of the distortion were increasing in T and approached infinity as $T \to \infty$ this would give a strong economic rational against indefinite taxation of capital. But in fact, we can show that the marginal cost remains bounded, even as $T \to \infty$.

We proceed with a constructive argument and assume the technology is linear for simplicity, so that $f(k, n) - \delta k = r^* k + w^* n$ for fixed $r^*, w^* > 0$.

Proposition 3.3

Suppose utility is given by (1.16), with $\sigma > 1$. Suppose technology is linear. Then the solution to the planning problem can be obtained by solving the following static problem:

$$\max_{T,c,n} u(c) - v(n)$$

s.t.

$$[1+\psi(T)]c + G = k_0 + \omega n,$$
$$u'(c)c - v'(n)n = [1-\tau(T)]u'(c)a_0,$$

where $\omega > 0$ is proportional to w^*; G is the present value of government consumption; and c and n are measures of lifetime consumption and labor supply. respectively. The functions $\psi(\cdot)$ and $\tau(\cdot)$ are increasing with $\phi(0) = \tau(0) = 0$; $\psi(\cdot)$ is bounded away from infinity and $\tau(\cdot)$ is bounded away from 1. Moreover, the marginal trade-off between costs ψ and benefits τ from extending capital taxation,

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{\psi'(T)}{\tau'(T)},$$

is bounded away from infinity.

Proof. First, we show that the planner's problem is equivalent to this static problem. Then we show that the functions $\psi(T)$ and $\tau(T)$ are increasing, have $\psi(0) = \tau(0) = 0$ and bounded derivatives.

The planner's problem in this linear economy can be written using a present value resource constraint, that is,

$$\max \int_0^\infty \mathrm{e}^{-\rho t}[u(c_t) - v(n_t)]\mathrm{d}t \tag{3.1}$$

s.t.

$$\dot{c} \geq c\frac{1}{\sigma}[(1-\bar{\tau})r^* - \rho], \tag{3.2}$$

$$\int_0^\infty \mathrm{e}^{-r^* t}(c_t - w^* n_t)\mathrm{d}t + G = k_0, \tag{3.3}$$

$$\int_0^\infty \mathrm{e}^{-\rho t}[(1-\sigma)u(c_t) - (1+\zeta)v(n_t)]\mathrm{d}t \geq u'(c_0)a_0, \tag{3.4}$$

where $G = \int_0^\infty \mathrm{e}^{-r^* t} g \mathrm{d}t$ is the present value of government expenses. Note that we assumed $\sigma > 1$.

Let Λ_1 and Λ_2 be the multiplier of constraint (3.3) and (3.4), respectively. Then the FOCs for labor supply can be written as

$$\frac{\partial}{\partial n_t}\int_0^\infty [-\mathrm{e}^{-\rho t}v(n_t) + \Lambda_1 \mathrm{e}^{-r^* t}w^* n_t - \Lambda_2 \mathrm{e}^{-\rho t}(1+\zeta)v(n_t)]\mathrm{d}t = 0,$$

that is,

$$-\mathrm{e}^{-\rho t}v'(n_t) + \Lambda_1 \mathrm{e}^{-r^* t}w^* - \Lambda_2 \mathrm{e}^{-\rho t}(1+\zeta)v'(n_t) = 0.$$

Solving for $v'(n_t)$ one yields

$$v'(n_t) = \frac{\Lambda_1 w^*}{1+\Lambda_2(1+\zeta)}\mathrm{e}^{-(r^*-\rho)t}.$$

Since $v'(n_t) = n_t^{\zeta}$, we solve for n_t that

$$n_t = \left[\frac{\Lambda_1 w^*}{1+\Lambda_2(1+\zeta)}\right]^{\frac{1}{\zeta}} \mathrm{e}^{\frac{-(r^*-\rho)t}{\zeta}}.$$

The initial condition is

$$n_0 = \left[\frac{\Lambda_1 w^*}{1+\Lambda_2(1+\zeta)}\right]^{\frac{1}{\zeta}}.$$

Thus,

$$n_t = n_0 \exp\left[-\frac{(r^*-\rho)}{\zeta} t\right], \tag{3.5}$$

given n_0.

By Euler equation (3.2), the general solution form is

$$c_t = c_0 \exp\left[\frac{(1-\bar{\tau})r^* - \rho}{\sigma} t\right].$$

The bang-bang result has shown that, $\exists T \in [0, \infty]$, such that the after-tax interest rate $r_t = (1-\bar{\tau})r^* \equiv \bar{r}$ for $t \leq T$, and $r_t = r^*$ for $t > T$. Then the path for consumption is determined by

$$c_t = c_0 \exp\left[-\frac{\rho - \bar{r}}{\sigma} t + \frac{r^* - \bar{r}}{\sigma}(t-T)^+\right]. \tag{3.6}$$

Since all the dynamic variables are pinned down, by substituting equations (3.5) and (3.6) into (3.3) and (3.4), the planner's problem can be simplified to a static problem,

$$\max_{T, c_0, n} \psi_1(T) u(c_0) - \psi_3 v(n_0)$$

s.t.

$$\psi_2(T)(\chi^*)^{-1} c_0 + G = k_0 + \psi_3 w^* n_0,$$
$$\psi_1(T) u'(c_0) c_0 - \psi_3 v'(n_0) n_0 = \chi^* u'(c_0) a_0,$$

where

$$\psi_1(T) = \frac{\chi^*}{\chi}(1 - \mathrm{e}^{-\chi T}) + \mathrm{e}^{-\chi T},$$
$$\psi_2(T) = \frac{\chi^*}{\hat{\chi}}(1 - \mathrm{e}^{-\hat{\chi} T}) + \mathrm{e}^{-\hat{\chi} T},$$
$$\psi_3 = \chi^* \left(r^* + \frac{r^* - \rho}{\zeta}\right)^{-1},$$

and

$$\chi = \frac{\sigma - 1}{\sigma}\bar{r} + \frac{\rho}{\sigma},$$
$$\chi^* = \frac{\sigma - 1}{\sigma} r^* + \frac{\rho}{\sigma},$$
$$\hat{\chi} = r^* + \frac{\rho - \bar{r}}{\sigma}.$$

We can rewrite the problem as

$$\max_{T, c, n} u(c) - v(n)$$

s.t.

$$
\begin{aligned}
[1+\psi(T)]c + G &= k_0 + \omega n, \\
u'(c)c - v'(n)n &= [1-\tau(T)]u'(c)a_0,
\end{aligned}
$$

by normalizing consumption and labor, c and n, respectively, and defining an efficiency cost $\psi(T)$, a capital levy $\tau(T)$, and the present value of wage ω,

$$
\begin{aligned}
c &\equiv \frac{\psi_1(T)^{\frac{1}{1-\sigma}}}{\chi^*} c_0, \\
n &\equiv \frac{\psi_3^{\frac{1}{1+\zeta}}}{(\chi^*)^{\frac{1-\sigma}{1+\zeta}}} n_0, \\
\psi(T) &\equiv \psi_2(T)\psi(T)^{\frac{1}{\sigma-1}} - 1, \\
\tau(T) &\equiv 1 - \psi_1(T)^{-\frac{\sigma}{\sigma-1}}, \\
\omega &\equiv w^* \psi_3^{\frac{\zeta}{1+\zeta}}.
\end{aligned}
$$

Here, by definition, ψ is bounded away from infinity and τ is bounded away from 1. Since $\psi_1(0) = \psi_2(0) = 1$, we obtain $\psi(0) = \tau(0) = 0$.

Now we discuss the monotonicity of these two functions. Firstly we have

$$
\psi_1'(T) = -\chi \left(1 - \frac{\chi^*}{\chi}\right) \mathrm{e}^{-\chi T} = (\chi^* - \chi)\mathrm{e}^{-\chi T}.
$$

Notice that $\hat{\chi} > \chi^* > \chi$. Then $\psi_1'(T) > 0$, $\forall T$. Since since $\sigma > 1$ and $\psi_1(T) \geq \psi_1(0) = 1$,

$$
\tau'(T) = \frac{\sigma}{\sigma - 1}\psi_1(T)^{\frac{1}{\sigma-1}}\psi_1'(T) > 0.
$$

These results show that $\psi_1(T)$ and $\tau(T)$ are increasing in T.

Similarly, we write $\psi_2'(T) = (\chi^* - \hat{\chi})\mathrm{e}^{-\hat{\chi}T} < 0$. Then, consider the derivative of ψ,

$$
\psi'(T) = \psi_2'(T)\psi_1(T)^{\frac{1}{\sigma-1}} + \psi_2(T) \cdot \frac{1}{\sigma-1}\psi_1(T)^{\frac{2-\sigma}{\sigma-1}}\psi_1'(T)
$$

Let $\psi'(T) \geq 0$. This requires that

$$
(\sigma-1)\frac{\psi_2'(T)}{\psi_1'(T)} \geq \frac{\psi_2(T)}{\psi_1(T)},
$$

that is,

$$
(\sigma-1)\frac{\chi^* - \hat{\chi}}{\chi^* - \chi}\mathrm{e}^{-(\hat{\chi}-\chi)T} \geq \frac{\chi}{\hat{\chi}}\frac{\chi^* - (\chi^* - \hat{\chi})\mathrm{e}^{-\hat{\chi}T}}{\chi^* - (\chi^* - \chi)\mathrm{e}^{-\chi T}}.
$$

On the LHS,

$$
(\sigma-1)\frac{\chi^* - \hat{\chi}}{\chi^* - \chi} = (\sigma-1)\frac{\bar{r} - r^*}{(\sigma-1)(r^* - \bar{r})} = -1,
$$

then the condition becomes

$$
-\mathrm{e}^{-(\hat{\chi}-\chi)T} \geq \frac{\chi}{\hat{\chi}}\frac{\chi^* - (\chi^* - \hat{\chi})\mathrm{e}^{-\hat{\chi}T}}{\chi^* - (\chi^* - \chi)\mathrm{e}^{-\chi T}}.
$$

By some rearrangement one yields

$$
\hat{\chi}\chi^*\mathrm{e}^{\chi T} - \hat{\chi}(\chi^* - \chi) \leq \chi\chi^*\mathrm{e}^{\hat{\chi}T} - \chi(\chi^* - \hat{\chi}).
$$

Finally, the equivalence condition of $\psi'(T) \geq 0$ is

$$\hat{\chi}(\mathrm{e}^{\chi T} - 1) \leq \chi(\mathrm{e}^{\hat{\chi} T} - 1),$$

which is true for any $T \geq 0$ because $\hat{\chi} > \chi$. Therefore, $\psi'(T) \geq 0$, with strict inequality for $T > 0$, implying that $\psi(T)$ is strictly increasing in T.

Now consider the ratio of derivatives,

$$\frac{\psi'(T)}{\tau'(T)} = \frac{(\sigma-1)\psi_2'\psi_1^{\frac{1}{\sigma-1}} + \psi_2\psi_1^{\frac{2-\sigma}{\sigma-1}}\psi_1'}{\sigma-1} \frac{\sigma-1}{\sigma\psi_1^{\frac{1}{\sigma-1}}\psi_1'} = \frac{1}{\sigma}\left[(\sigma-1)\frac{\psi_2'}{\psi_1'} + \frac{\psi_2}{\psi_1}\right].$$

Notice that $\psi_1(T) \in [1, \chi^*/\chi]$ and $\psi_2(T) \in [\chi^*/\hat{\chi}, 1]$, so both are bounded away from infinity and zero. Further, the ratio $\psi_2'/\psi_1' \in [-1/(\sigma-1), 0]$, is also bounded away from infinity, implying that $\psi'(T)/\tau'(T)$ is bounded away from infinity, too. $\square$

Since the trade-off between costs and benefits is bounded, $\mathrm{d}\psi/\mathrm{d}\tau < \infty$, even as $T \to \infty$, the indefinite taxation does not come at an infinite marginal cost.

Effectively, a bound on capital taxation restricts the path for the consumption tax to lie below a straight line going through the origin. In the short run, the consumption tax is constrained to be near zero; to compensate, it is optimal to set higher consumption taxes in the future. As a result, it may be optimal to set consumption taxes as high as possible at all times. This is equivalent to indefinite capital taxation.