Reading Notes of Straub & Werning (2020) “Positive Long-Run Capital Taxation” Part 1. Two-Agent Model with Incomplete Market

George Chou

May 27, 2026

1 Environment

1.1 Preferences

Capitalists do not work, and have objective function

$$\sum_{t=0}^{\infty} \beta^t U(C_t),$$

with a isoelastic (σ^{-1}) utility, where C_t denotes the consumption by capitalists.

Workers have a constant labor endowment $n = 1$, and objective function

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where c_t denotes the consumption by workers, u is increasing, concave, continuously differentiable, and $\lim_{c\to 0} u'(c) = \infty$.

1.2 Technology

Output is obtained from capital and labor using a neoclassical CRS production function $F(k_t, n_t)$ satisfying standard conditions. In equilibrium $n_t = n = 1$, so define $f(k) \equiv F(k, 1)$. The populations is normalized to unity, and technological progress and population growth are abstracted.

The resource constraint in period t is then

$$c_t + C_t + g + k_{t+1} \leq f(k_t) + (1-\delta)k_t, \tag{1.1 RC}$$

where $g > 0$ denotes the constant government consumption, $\delta \in [0, 1]$ denotes the depreciate rate of capital, and the initial capital $k_0 > 0$ is given.

1.3 Markets and Taxes

Markets are perfectly competitive, with labor being paid wage $w_t^* = F_n(k_t, n_t)$ and the before-tax return on capital being given by

$$R_t^* = f'(k_t) + 1 - \delta. \tag{1.2}$$

The after-tax return can be represented as

$$R_t = (1-\tau_t)(R_t^* - 1) + 1 = (1-\mathcal{T}_t)R_t^*, \tag{1.3}$$

where τ_t is the tax rate on the net return to wealth, and $\mathcal{T}_t$ is the (gross) wealth tax rate.

1.4 Agents' Behavior

The UMP of capitalists:

$$\max_{\{C_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t)$$

s.t.

$$C_t + a_{t+1} = R_t a_t,$$
$$a_{t+1} \geq 0,$$

for some given initial wealth $a_0 > 0$. Then the Euler equation,

$$\frac{U'(C_t)}{U'(C_{t+1})} = \beta R_{t+1}, \tag{1.4}$$

and transversality condition,

$$\lim_{t \to \infty} \beta^t U'(C_t) a_{t+1} = 0, \tag{1.5}$$

are necessary and sufficient for optimality.

Workers live hand-to-mouth, their consumption equals their disposable income

$$c_t = w_t^* + T_t \xlongequal{F_n = F - F_k k} f(k_t) - f'(k_t)k_t + T_t, \tag{1.6}$$

where $T_t \in \mathbb{R}$ represent government lump-sum transfers / taxes to workers.

1.5 Government Budget Constraint

The government cannot issue bonds and runs a balanced budget. This implies that total wealth equals the capital stock $a_k = k_t$, and that the government budget constraint is

$$g + T_t = (R_t^* - R_t) k_t. \tag{1.7 GBC}$$

1.6 Planning Problem

Combining the capitalists' budget constraint and equation (1.6) we obtain

$$\frac{U'(C_{t-1})}{U'(C_t)} = \beta \frac{C_t + k_{t+1}}{k_t}. \tag{1.8 ICC}$$

Equation (1.4) can also be represented as

$$\lim_{t \to \infty} \beta^t U'(C_t) k_{t+1} = 0. \tag{1.9 TVC}$$

The government maximizes a weighted sum of utilities with γ on capitalists. Then the planning problem can be written as

$$\max_{C_{-1}, \{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(c_t) + \gamma U(C_t)] \tag{SWF}$$

s.t. equation (1.1 RC), (1.8 ICC), and (1.9 TVC).

Define the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t) + \gamma U(C_t)] + \sum_{t=0}^{\infty} \beta^t \lambda_t [f(k_t) + (1 + \delta)k_t - (c_t + C_t + g + k_{t+1})]$$
$$+ \sum_{t=0}^{\infty} \beta^t \mu_t [\beta U'(C_t)(C_t + k_{t+1}) - U'(C_{t-1})k_t].$$

Now we start to derive the necessary FOCs: equation (1.10), (1.11), (1.12), and (1.13). The derivative w.r.t. C_{-1}:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_{-1}} &= \sum_{t=0}^{\infty} \beta^t \mu_t \cdot \frac{\partial}{\partial C_{-1}}[\beta U'(C_t)(C_t + k_{t+1}) - U'(C_{t-1})k_t] \\ &= \beta^0 \mu_0 [0 - U''(C_{-1})k_0] \\ &= -\mu_0 U''(C_{-1})k_0 = 0.\end{aligned}$$

Since $U''(\cdot) < 0$ and $k_0 > 0$, we have

$$\mu_0 = 0. \tag{1.10}$$

The derivative w.r.t. c_t:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \beta^t - \lambda_t = 0,$$

that is,

$$u'(c_t) = \lambda_t. \tag{1.11}$$

The derivative w.r.t. C_t:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \gamma U'(C_t) - \beta^t \lambda_t + \beta^t \mu_t \cdot \beta[U''(C_t)(C_t + k_{t+1}) + U'(C_t)] + \beta^{t+1}\mu_{t+1}[-U''(C_t)k_{t+1}] = 0.$$

Dividing by β^t and using equation (1.11) one yields

$$\begin{aligned} 0 &= \gamma U'(C_t) - u'(c_t) + \beta\mu_t[U''(C_t)(C_t + k_{t+1}) + U'(C_t)] - \beta\mu_{t+1}U''(C_t)k_{t+1} \\ &\overset{v_t := \frac{U'(C_t)}{u'(c_t)}}{=\!=\!=} \gamma v_t u'(c_t) - u'(c_t) + \beta\mu_t[U''(C_t)(C_t + k_{t+1}) + v_t u'(c_t)] - \beta\mu_{t+1}U''(C_t)k_{t+1} \\ &\overset{\sigma = -\frac{U'(C_t)}{U''(C_t)C_t}}{=\!=\!=} \gamma v_t u'(c_t) - u'(c_t) + \beta\mu_t\left[-\frac{U'(C_t)}{\sigma C_t}(C_t + k_{t+1}) + v_t u'(c_t)\right] + \beta\mu_{t+1}\frac{U'(C_t)}{\sigma C_t}k_{t+1} \\ &= \gamma v_t u'(c_t) - u'(c_t) + \beta\mu_t\left[-\frac{v_t u'(c_t)}{\sigma C_t}(C_t + k_{t+1}) + v_t u'(c_t)\right] + \beta\mu_{t+1}\frac{v_t u'(c_t)}{\sigma C_t}k_{t+1}. \end{aligned}$$

Dividing by $u'(c_t)$ one yields

$$\begin{aligned} 0 &= \gamma v_t - 1 + \beta\mu_t\left[-\frac{v_t}{\sigma}\left(1 + \frac{k_{t+1}}{C_t}\right) + v_t\right] + \beta\mu_{t+1}\frac{v_t}{\sigma}\frac{k_{t+1}}{C_t} \\ &\overset{\kappa_t := \frac{k_t}{C_{t-1}}}{=\!=\!=} \gamma v_t - 1 + \beta\mu_t v_t\left[1 - \frac{1 + \kappa_{t+1}}{\sigma}\right] + \beta\mu_{t+1}\frac{v_t \kappa_{t+1}}{\sigma}. \end{aligned}$$

Solving for the law of motion of μ_t we have

$$\mu_{t+1} = \mu_t\left(\frac{\sigma - 1}{\sigma\kappa_{t+1}} + 1\right) + \frac{1}{\beta\sigma\kappa_{t+1}v_t}(1 - \gamma v_t). \tag{1.12}$$

The derivative of k_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\beta^t\lambda_t + \beta^{t+1}\lambda_{t+1}[f'(k_{t+1}) + 1 - \delta] + \beta^t\mu_t \cdot \beta U'(C_t) - \beta^{t+1}\mu_{t+1}U'(C_t) = 0.$$

Dividing by $\beta^t\lambda_t$ and using equation (1.11) one yields

$$\begin{aligned} 0 &= 1 + \beta\frac{\lambda_{t+1}}{\lambda_t}[f'(k_{t+1}) + 1 - \delta] + \beta(\mu_t - \mu_{t+1})\frac{U'(C_t)}{\lambda_t} \\ &= 1 + \beta\frac{u'(c_{t+1})}{u'(c_t)}[f'(k_{t+1}) + 1 - \delta] + \beta(\mu_t - \mu_{t+1})\frac{U'(C_t)}{u'(c_t)} \\ &= 1 + \beta\frac{u'(c_{t+1})}{u'(c_t)}[f'(k_{t+1}) + 1 - \delta] + \beta(\mu_t - \mu_{t+1})v_t. \end{aligned}$$

Then we obtain

$$\frac{u'(c_{t+1})}{u'(c_t)}[f'(k_{t+1})+1-\delta] = \frac{1}{\beta} + v_t(\mu_{t+1}-\mu_t). \tag{1.13}$$

2 Previous Results by Judd (1985)

Judd (1985) provided a zero-tax result, which we adjust in the following statement to stress the need for the steady state to be interior and for multipliers to converge.

Theorem 2.1: Judd (1985)

Suppose quantities and multipliers converge to an interior steady state, i.e., c_t, C_t, k_{t+1} converge to positive values, and μ_t converges. Then the tax on capital is zero in the limit: $\mathcal{T}_t \to 0$.

Proof. Taking the limit for $t \to \infty$ in equation (1.4), we obtain

$$\lim_{t\to\infty} \frac{U'(C_t)}{U'(C_{t+1})} = 1 = \beta \lim_{t\to\infty} R_{t+1}.$$

Thus, $\lim_{t\to\infty} R_t = \beta^{-1}$.
Taking the limit for $t \to \infty$ in equation (1.13), we obtain

$$\text{LHS} = \lim_{t\to\infty} \frac{u'(c_{t+1})}{u'(c_t)}[f'(k_{t+1})+1-\delta] \xlongequal{R_t^*=f'(k_t)+1-\delta} \lim_{t\to\infty} \frac{u'(c_{t+1})}{u'(c_t)} R_t^* = \lim_{t\to\infty} R_t^*,$$

and

$$\text{RHS} = \beta^{-1} + \lim_{t\to\infty} v_t(\mu_{t+1}-\mu_t) = \beta^{-1}.$$

Then we have $\lim_{t\to\infty} R_t^* = \beta^{-1}$. Hence,

$$\lim_{t\to\infty} \mathcal{T}_t = 1 - \lim_{t\to\infty} \frac{R_t}{R_t^*} = 0.$$

□

The two situations that prevent the theorem's application are

(i) Nonconvergence to an interior steady state. In other words, $u'(c_{t+1})/u'(c_t)$ do not converge to 1 when $c_t \to 0$; or

(ii) Nonconvergence of $\mu_{t+1} - \mu_t$ to zero.

We give a counterexample of the second case where the allocation does converge to an interior steady state, which is discussed by Lansing (1999) and Reinhorn (2019).

Theorem 2.2: Lansing (1999) & Reinhorn (2019)

Assume $\sigma = 1$, i.e., $U(C_t) = \ln C_t$. Suppose the allocation converges to an interior steady state, so that $c_t, C_t k_{t+1}$ converge to strictly positive values. Then

$$\mathcal{T}_t \to \frac{1-\beta}{1+\frac{\gamma v \beta}{1-\gamma v}},$$

where $v = \lim v_t$, and the multiplier μ_t does not converge. This implies a positive long-run tax on capital if redistribution toward workers is desirable, $1 - \gamma v > 0$.

Proof. Omitted. □

3 Revisiting Judd (1985)

3.1 Logarithmic Utility

Proposition 3.1

Suppose $\gamma = 0$ and that capitalists have logarithmic utility, $U(C) = \ln C$, i.e., $\sigma = 1$. Then the solution to the planning problem converges monotonically to a unique steady state with a positive tax on capital given by $\mathcal{T} = 1 - \beta$.

Proof. When $\sigma = 1$, capitalists save at a constant rate $s > 0$ (indeed, $s = \beta$ in this case), then we have $C_t = (1-s)R_k k_t$ and $k_{t+1} = sR_t k_t$.

Without the EE and TVC of capitalists' optimization problem, the planning problem can be simplified as

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c_t + s^{-1} k_{t+1} + g = f(k_t) + (1-\delta)k_t,$$

where k_0 is given.

This amounts to an optimal neoclassical growth problem, and we know that there exists a unique interior steady state and that it is globally stable. The modified golden rule at this steady state is $\beta s R^* = 1$. A steady state also requires $sR = 1$. Putting these conditions together gives $\mathcal{T} = 1 - R/R^* = 1 - \beta > 0$. □

3.2 Positive Long-Run Taxation: EIS < 1

Proposition 3.2

If $\sigma > 1$ and $\gamma = 0$, no solution to the planning problem converges to the zero-tax steady state, or any other interior steady state.

Proof. Toward a contradiction, suppose there existed an optimal allocation that converges to an interior steady state $k_t \to k$, $C_t \to C$, $c_t \to c$ with $k, C, c > 0$. This implies $\kappa_t \to \kappa$ and $v_t \to v$ with $\kappa, v > 0$. Moreover, the entire path $\{k_t, C_{t-1}, c_t\}$ must also be interior, such that the four FOCs necessarily hold at the optimum.

Taking the limit for equation (1.12), we obtain

$$\lim_{t\to\infty} (\mu_{t+1} - \mu_t) = \frac{\sigma - 1}{\sigma\kappa} \lim_{t\to\infty} \mu_t + \frac{1 - \gamma v}{\beta\sigma\kappa v}.$$

Taking the limit for equation (1.13) then using the equation above, we have

$$R = f'(k) + 1 - \delta = \frac{1}{\beta} + \frac{v(\sigma-1)}{\sigma\kappa} \lim_{t\to\infty} \mu_t + \frac{1}{\beta\sigma\kappa}.$$

Equation (1.4) requires $\beta R = 1$. Since $\sigma > 1$, this means that μ_t must converge to

$$\mu = -\frac{1}{(\sigma-1)\beta v} < 0.$$

Now consider whether $\mu_t \to \mu < 0$ is possible. From equation (1.10) we have $\mu_0 = 0$. Also, from equation (1.12), whenever $\mu_t \geq 0$ then $\mu_t \geq 0$. It follows that $\mu_t \geq 0$, $\forall t = 0, 1, \ldots$, a contradiction to $\mu_t \to \mu < 0$.

This proves that the solution cannot converge to any interior steady state, including the zero-tax steady state. □

Corollary 3.3

If the optimal allocation converges, then $c_t \to 0$.

Proof. If the optimal allocation converges, then either $k_t \to 0$, $C_t \to 0$, or $c_t \to 0$. With positive spending $g > 0$, by the resource constraint (1.1 RC), $k_t \to 0$ is not feasible. Since capitalists cannot be starved while owning positive wealth, $C_t \to 0$ is also ruled out. Thus, There must be $c_t \to 0$. □

Corollary 3.4

If the optimal allocation converges, then either $k_t \to \overline{k}$ or $k_t \to \underline{k}$, where the lowest / highest sustainable value of capital, $\underline{k}$ and $\overline{k}$, respectively, are the two solutions to $k/\beta + g = f(k) + (1-\delta)k$.

Proof. At steady state, the equation (1.8 ICC) can be written as $\beta(C+k) = k$ or $C + k = k/\beta$. By Corollary 3.3, we have $c_t \to 0$. Then using the limiting form of equation (1.1 RC) we have $k/\beta + g = f(k) + (1-\delta)k$. Thus, k_t must converge to the value which is the solution of this equation. □

Proposition 3.5

If $\sigma > 1$ and $\gamma = 0$, any solution to the planning problem converges to $c_t \to 0$, $k_t \to \underline{k}$, $C_t \to \frac{1-\beta}{\beta}\underline{k}$, with a positive limit tax on wealth: $\mathcal{T}_t \to \overline{\mathcal{T}} > 0$. The limit tax $\overline{\mathcal{T}}$ is decreasing in spending g, with $\overline{\mathcal{T}} \to 1$ as $g \to 0$.

Proof. By Lemma 4.11 and Lemma 4.14, we have shown that the planning problem converges to $c_t \to 0$, $k_t \to \underline{k}$, and $C_t \to \frac{1-\beta}{\beta}\underline{k}$. By equation (4.4), $F'(k) = f'(k) + 1 - \delta = R^*$. When $k = \underline{k}$, $F'(\underline{k}) > \beta^{-1} = R$. Thus, $\mathcal{T} = 1 - R/R^* > 0$.

Since $F(\underline{k}) - g = \beta^{-1}\underline{k}$, taking the derivative w.r.t g,

$$[f'(\underline{k}) + 1 - \delta - \beta^{-1}]\frac{\mathrm{d}\underline{k}}{\mathrm{d}g} - 1 = 0,$$

that is,

$$\frac{\mathrm{d}\underline{k}}{\mathrm{d}g} = \frac{1}{f'(\underline{k}) + 1 - \delta - \beta^{-1}} = \frac{1}{F'(\underline{k}) - \beta^{-1}} > 0.$$

This proves that $\overline{\mathcal{T}}$ is decreasing in g.

Moreover, when $g = 0$, equation (4.4) becomes

$$f(\underline{k}) + (1-\delta)\underline{k} = \beta^{-1}\underline{k}.$$

Since $f(0) = 0$ and $f'(0) = \infty$, $\underline{k} = 0$ is definitely a solution. Then by continuity, when $g \to 0$, $R^* = f(\underline{k}) + 1 - \delta \to \infty$, and $\overline{\mathcal{T}} \to 1$. □

3.3 Numerical Solution for EIS Near 1

For numerical computing, we use a recursive representation of the planning problem. The two constraints (1.1 RC) and (1.8 ICC) in the problem feature the variables C_{t-1}, k_t, C_t, k_{t+1}, and c_t. This suggests a recursive formulation with (k_t, C_{t-1}) as the state and c_t as a control.

The associated Bellman equation is then

$$V(k, C_-) = \max_{c \geq 0, (k', C) \in A} u(c) + \gamma U(C) + \beta V(k', C) \tag{3.1}$$

s.t.

$$c + C + k' + g = f(k) + (1 - \delta)k, \tag{3.2}$$

$$\beta U'(C)(C + k') = U'(C_-)k, \tag{3.3}$$

$$c, C, k' \geq 0, \tag{3.4}$$

where A is the feasible set, that is, states (k_0, C_{-1}) such that there exists a sequence $\{k_{t+1}, C_t\}$ satisfying all the constraints in the programming problem including the TVC.

At $t = 0$, k_0 is given, so there is no need to impose $\beta U'(C_0)(C_0 + k_1) = U'(C_{-1})k_0$. Thus, the planner maximizes $V(k_0, C_{-1})$ w.r.t. C_{-1}. If V is differentiable, the FOC is

$$V_C(k_0, C_{-1}) = 0.$$

Since one can show that $\mu_t = V_C(k_t, C_{t-1})U''(C_{t-1})k_t$, this is akin to the condition $\mu_0 = 0$ in equation (1.10).

Then using the value function iteration, we obtain the dynamic path of $\{k_t\}$ and $\{\mathcal{T}_t\}$, which is plotted in these figures below[1]:

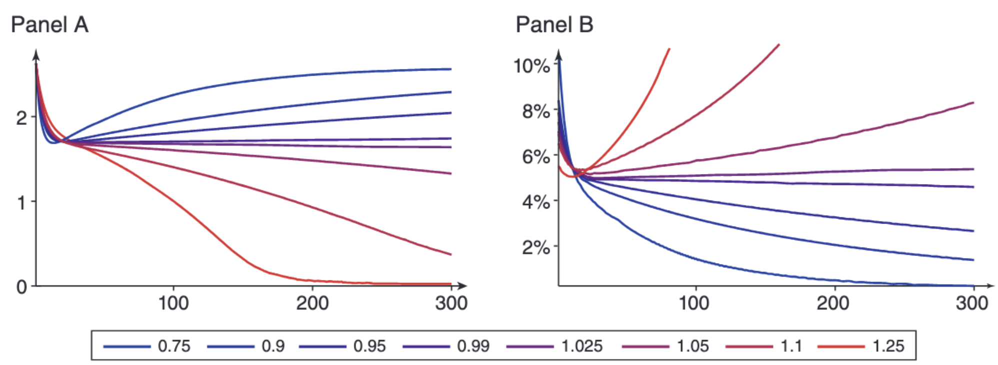


FIGURE 1. OPTIMAL TIME PATHS FOR CAPITAL (PANEL A) AND WEALTH TAXES (PANEL B)

Note: This figure shows the optimal time paths of capital k_t (panel A) and wealth taxes $\mathcal{T}_t$ (panel B) for various values of the inverse IES σ.

4 Full Proof of Proposition 3.5

4.1 Preliminaries

Let $x = k' + C$, then equation (3.3) is expressed as $x = \frac{U'(C_-)}{U'(C)} \frac{k}{\beta}$. Solving for C we have

$$C = C_- \left(\frac{\beta x}{k} \right)^{\frac{1}{\sigma}}, \tag{4.1}$$

then

$$k = x - C = x - C_- \left(\frac{\beta x}{k} \right)^{\frac{1}{\sigma}}. \tag{4.2}$$

[1]We set $\beta = 0.95$, $\delta = 0.1$, $f(k) = k^\alpha$ with $\alpha = 0.3$, and $u(c) = U(c)$. g is chosen so that $g/f(k) = 20\%$ at the zero-tax steady state. k_0 is set at the zero-tax steady state.

Substituting out x, we can also write the law of motion for capital as $k' = \frac{1}{\beta}\frac{k}{C_-^{\sigma}}C^{\sigma} - C$.

Since $c \geq 0$, by equation (3.2) we have $x \leq f(k) + (1-\delta)k - g$. Similarly, in equation (4.2) we have $x \geq C_- \left(\frac{\beta x}{k}\right)^{\frac{1}{\sigma}}$, that is, $x \geq C_-^{\frac{\sigma}{\sigma-1}}\left(\frac{\beta}{k}\right)^{\frac{1}{\sigma-1}}$ due to $k' \geq 0$. In sum, we obtain

$$C_-^{\frac{\sigma}{\sigma-1}}\left(\frac{\beta}{k}\right)^{\frac{1}{\sigma-1}} \leq x \leq f(k) + (1-\delta)k - g. \tag{4.3}$$

In recursive statement, a state (k, C_-) can be abbreviated by z, with the index set Z. The total space is denoted by Z_{all}. The whole set of future states z' which can follow a given state z is denoted by $\Gamma(z)$, which can be the empty set.

Now we discuss about the strict definition of feasible set A in Bellman equation.

Definition 4.1: Feasible Path

We call a path $\{z_t\}$ feasible, if $z_{t+1} \in \Gamma(z_t)$, $\forall t \geq 0$ (which precludes $\Gamma(z_t) = \emptyset$), and the TVC $\beta^t C_t^{-\sigma} k_{t+1} \to 0$ holds along the path.

Definition 4.2: Feasible State

We call a state z feasible, if there exist a feasible (infinite) path $\{z_t\}$ starting at $z_0 = z$. The set of all feasible states is denoted by Z^*.

Definition 4.3: Self-Generating

We say z is generated by $\{z_t\}$ when z is feasible. Because $z_1 \in \Gamma(z)$, we also say z is generated by z_1. A set Z is called self-generating if $\forall z \in Z$ is generated by a sequence in Z.

It will be important to specify between which capital stocks the economy is moving. Motivated by Corollary 3.4, define $\underline{k}$ and $\overline{k} > \underline{k}$ to be the two roots to the equation[2]

$$k = F(k) - \frac{1-\beta}{\beta}k, \tag{4.4}$$

where $F(k) := f(k) + (1-\delta)k - g$. Demanding that $\overline{k} > \underline{k}$ is tantamount to specifying $F'(\overline{k}) < \beta^{-1} < F'(\underline{k})$.

Corresponding to $\overline{k}$ and $\underline{k}$, we define $\overline{C} := \frac{1-\beta}{\beta}\overline{k}$ and $\underline{C} := \frac{1-\beta}{\beta}\underline{k}$ as the respective steady state consumption of capitalists. The steady states $(\underline{k}, \underline{C})$ and $(\overline{k}, \overline{C})$ represent the lowest and highest feasible steady states, respectively, in which reason is that $\forall k \notin [\underline{k}, \overline{k}]$, the steady state RC (4.4) is violated.

Unlike the neoclassical growth model, our system do not allow for arbitrarily large capital stocks. To implement a upper bound of capital, we take the total state space to be $Z_{\text{all}} = [0, k_{\max}] \times \mathbb{R}_+$ for $z = (k, C_-)$, where $k_{\max} = \max\{\tilde{k}, k_0\}$ and $k = \tilde{k}$ solves $k = F(k)$. Then the set of capital stocks that are resource feasible given k_0 must necessarily lie in $[0, k_{\max}]$. Notice that with this state space, the set of feasible states Z^* is also capped at $k_{\max}$ in its k-component.

4.2 Geometry of Feasible Set

In this subsection, we manage to characterize the bottom and top boundaries of feasible set Z^*, by splitting up the total space $Z_{\text{all}} = [0, k_{\max}] \times \mathbb{R}_+$ into four pieces and characterizing

[2]If there are less then two roots of this equation, then this implies that government consumption g is unsustainably high for any capital stock, and such values for g are uninteresting and therefore ruled out.

the feasible states in each of the four pieces, which is illustrated by the figure below.

Define

$$\underline{w}(k) := \begin{cases} \frac{1-\beta}{\beta}k & k \in [0, \underline{k}] \\ \underline{C}\left(\frac{k}{\underline{k}}\right)^{\frac{1}{\sigma}} & k \in (\underline{k}, k_{\max}] \end{cases}$$

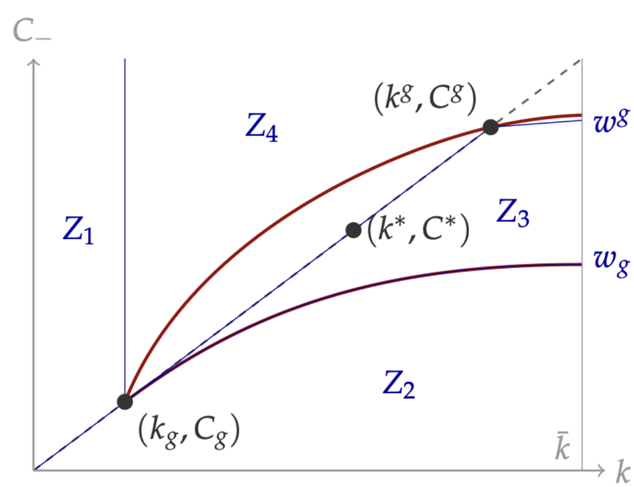
$$\overline{w}(k) := \begin{cases} \frac{1-\beta}{\beta}k & k \in [0, \underline{k}] \\ \overline{C}\left(\frac{k}{\overline{k}}\right)^{\frac{1}{\sigma}} & k \in (\overline{k}, k_{\max}] \end{cases}$$

and split up the total state space as follows

$$Z_{\text{all}} = Z_1 \cup Z_2 \cup Z_3 \cup Z_4,$$

where

$$\begin{aligned} Z_1 &= \{k < \underline{k}, C_- \geq \underline{w}(k)\}, \\ Z_2 &= \{C_- < \underline{w}(k)\}, \\ Z_3 &= \{k \geq \overline{k}, \underline{w}(k) \leq C_- \leq \overline{w}(k)\}, \\ Z_4 &= \{k \geq \underline{k}, C_- \geq \overline{w}(k)\}. \end{aligned}$$



The next two lemmas characterize the lower boundary of Z^*[3].

Lemma 4.4

$Z^* \cap Z_1 = Z^* \cap Z_2 = \emptyset$. All states with $k < \underline{k}$ or $C_- < \underline{w}(k)$ are infeasible.

Lemma 4.5

$Z_3 \subset Z^*$, all states with $\underline{w}(k) \leq C_- \leq \overline{w}(k)$ and $k \geq \underline{k}$ are feasible and generated by a feasible steady state. Moreover, states on the boundary $\{C_- = \underline{w}(k), k > \underline{k}\}$ can only be generated by a single feasible state, $(\underline{k}, \underline{C})$. Thus, there is only a single "feasible" control for those states, $c > 0$.

Then we turn to characterize the top boundary. By iterative construction, we can show that the boundary takes the form of an increasing function $\hat{w}(k)$ such that states with $C_- > \overline{w}(k)$ are feasible iff $C_- \leq \hat{w}(k)$.

[3]All proofs in this subsection are not given.

Abbreviating the laws of motion for capitalists' consumption and capital, equations (4.1) and (4.2), as $k'(x, k, C_-)$ and $C(x, k, C_-)$. Define an operator T on the space of continuous, increasing functions $v : [\underline{k}, k_{\max}] \to \mathbb{R}_+$, as,

$$Tv(k) = \sup\{C_- | \exists x \in (0, F(k)) : v(k'(\cdot)) \geq C(\cdot)\}.$$

The operator T is designed to extend a candidate top boundary of the set of feasible states by one iteration.

The next two lemmas characterize the limit function $\hat{w}(k)$, whose graph will describe the top boundary of Z^*.

Lemma 4.6

There exists a continuous limit function $\hat{w}(k) := \lim_{i\to\infty} T^i \overline{w}(k) = T\hat{w}(k)$, with $\hat{w}(\underline{k}) = \underline{C}$ and $\hat{w}(\overline{k}) = \overline{C}$. All states with $C_- = \hat{w}(k)$ are feasible, but only with policy $c = 0$.

Lemma 4.7

No state with $C_- > \hat{w}(k)$ and $\underline{k} \leq k \leq k_{\max}$ is feasible.

In sum, the feasible set Z^* can be represented as

$$Z^* = [\underline{k}, k_{\max}] \times \Big\{\{C_- | \underline{w}(k) \leq C_- \leq \overline{w}(k)\} \cup \{C_- | C_- = \hat{w}(k)\}\Big\}.$$

Finally, we give another lemma which is used in the following proof.

Lemma 4.8

Let $\{k_{t+1}, C_t\}$ be the path starting at (k_0, C_{-1}) with controls $c_t = 0$. Let $\underline{k} < k_0 \leq k_{\max}$. Then:

1. If $C_{-1} = \hat{w}(k_0)$, $(k_{t+1}, C_t) \to (\overline{k}, \overline{C})$.
2. If $C_{-1} > \hat{w}(k_0)$, $(k_{t+1}, C_t) \nrightarrow (\overline{k}, \overline{C})$.

4.3 Formal Proof

For all statements in this subsection, we only consider the economy with an initial capital stock of $k_0 \in [\underline{k}, k_{\max}]$.

Definition 4.9: Optimal Path

We call a path $\{k_{t+1}, C_t\}$ optimal path, if the initial C_{-1} was optimized over given k_0.

Definition 4.10: Locally Optimal Path

We call a path $\{k_{t+1}, C_t\}$ locally optimal path, if the initial C_{-1} was not optimized over, but rather taken as given at a certain level, respecting the constraint that (k_0, C_{-1}) be feasible. In this case, we say c_{t+1} optimal control at $\{k_{t+1}, C_t\}$.

The first lemma proves that the multiplier on the capitalists' IC constraint μ_t explodes, and worker's consumption c_t drops to zero, along an optimal path.

Lemma 4.11

Along any optimal path, $c_t \to 0$.

Proof. Let $\{k_{t+1,C_t}\}$ be the optimal path. Suppose first the optimal path hits the boundary of the feasible set Z^* at some finite time. Given that, no path can hit the $k = k_{\max}$ boundary after $t = 0$. And given Lemma 4.5, the path hits the top boundary after finite time. Lemma 4.6 showed that along that boundary, the control is necessarily zero, $c = 0$.

Now suppose the optimal path is interior at all times. In that case, the first order conditions are necessary. In particular, The one for μ_t (1.12) states ($\gamma = 0$)

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \kappa_{t+1} v_t}.$$

From Lemma 4.2 we know that $\kappa_{t+1} = \frac{k_{t+1}}{C_t} < \infty$. Since $\mu_0 = 0$ by (1.10) and $\sigma > 1$, it follows that $\mu_t \geq 0$ and it will diverge.

Toward a contraction, suppose that $c_t \not\to 0$. In this case, there exists $c_{\min} > 0$ and an infinite sequence of time indices (t_s) such that $c_{t_s} \geq c_{\min}$, $\forall s$. Along these indices the FOC for capital (1.13) implies

$$u'(c_{t_s})[f'(k_{t_s}) + (1 - \delta)] = \frac{1}{\beta} u'(c_{t_s - 1}) + U'(C_{t_s - 1})(\mu_{t_s} - \mu_{t_s - 1}).$$

In the RHS, $u'(c_{t_s-1})$ and $U'(C_{t_s-1})$ are both positive, and $(\mu_{t_s} - \mu_{t_s-1}) \to \infty$, so that RHS $\to \infty$. But in the LHS, $u'(c_{t_s}) \leq u'(c_{\min})$ is finite, which implies $f'(k_{t_s}) \to \infty$, i.e., $k_{t_s} \to 0$ for $s \to \infty$. This is impossible within the feasible set Z^*. This proves that also for interior optimal paths, $c_t \to 0$. □

With the following lemma, these two results gives us a crucial geometric restriction of where an optimal path goes in the long run.

Lemma 4.12

The set of states where $c = 0$ is an optimal control is the top boundary of Z^*. It follows that an optimal path approaches either $(\underline{k}, \underline{C})$ or $(\overline{k}, \overline{C})$.

Proof. Omitted. □

By showing that the optimal path cannot converge to one of them, finally we proves Proposition 3.5.

Lemma 4.13

If an optimal path $\{k_{t+1}, C_t\}$ converges to $(\overline{k}, \overline{C})$, then the value function V is locally decreasing in C at each point (k_{t+1}, C_t), $\forall t > T$, with T large enough.

Proof. Let $x_t \equiv F(k_t) - c_t$ and consider the following variation: Suppose that at time point T, (k_{T+1}, C_T) cannot converge to $(\overline{k}, \overline{C})$ anyway, and that $c_t < F(k_t) - F'(k_t)k_t$, $\forall t \; geT$[4]. For simplicity, call this $T = -1$.

[4]Such a finite $T > 0$ exists, because $c_t \to 0$ and $F(k) - F'(k)k > 0$ in a neighborhood around $k = \overline{k}$.

Then we do the perturbation $\hat{C}_{-1} \equiv C_{-1} - \varepsilon$ $(\varepsilon > 0)$, $\hat{k}_0 = k_0$, and $\hat{c}_t = c_t$. Denote the perturbed capital stock and capitalists' consumption by $\hat{k}_{t+1} = k_{t+1} + \mathrm{d}k_{t+1}$ and $\hat{C}_t = C_t + \mathrm{d}C_t$, Then x changes by $\mathrm{d}x = F'(k_t)\mathrm{d}k_t$ to first order.

Firstly we consider the local law of motion of C_t. Taking the logarithm of equation (4.1) one obtains

$$\ln C_t = \ln C_{t-1} + \frac{1}{\sigma}(\ln \beta + \ln x - \ln k).$$

Taking the differential on both sides,

$$\begin{aligned}
\frac{\mathrm{d}C_t}{C_t} &= \frac{\mathrm{d}C_{t-1}}{C_{t-1}} + \frac{1}{\sigma}\left(\frac{\mathrm{d}x}{x} - \frac{\mathrm{d}k}{k}\right) \\
&= \frac{\mathrm{d}C_{t-1}}{C_{t-1}} - \frac{1}{\sigma}\frac{1}{x_t k_t}(x_t \mathrm{d}k_t - k_t \mathrm{d}x_t) \\
&= \frac{\mathrm{d}C_{t-1}}{C_{t-1}} - \frac{1}{\sigma}\frac{1}{x_t k_t}[(F(k_t) - c_t)\mathrm{d}k_t - F'(k_t)k_t \mathrm{d}k_t] \\
&= \frac{\mathrm{d}C_{t-1}}{C_{t-1}} - \frac{1}{\sigma}\frac{1}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}\mathrm{d}k_t.
\end{aligned}$$

Multiplying by C_t, we have

$$\mathrm{d}C_t = \frac{C_t}{C_{t-1}}\mathrm{d}C_{t-1} - \frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}\mathrm{d}k_t.$$

Then similarly to k_{t+1}, we have

$$\mathrm{d}k_{t+1} = \mathrm{d}x_t - \mathrm{d}C_t = F'(k_t)\mathrm{d}k_t - \frac{C_t}{C_{t-1}}\mathrm{d}C_{t-1} + \frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}\mathrm{d}k_t.$$

Using matrix notation, this local low of motion can be written as

$$\begin{bmatrix} \mathrm{d}k_{t+1} \\ \mathrm{d}C_t \end{bmatrix} = \begin{bmatrix} F'(k_t) + \frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t} & -\frac{C_t}{C_{t-1}} \\ -\frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t} & \frac{C_t}{C_{t-1}} \end{bmatrix} \begin{bmatrix} \mathrm{d}k_t \\ \mathrm{d}C_{t-1} \end{bmatrix}. \tag{4.5}$$

Denote $M_t = \frac{C_t}{C_{t-1}}$ and $N_t = \frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}$, then the system becomes

$$\begin{bmatrix} \mathrm{d}k_{t+1} \\ \mathrm{d}C_t \end{bmatrix} = \begin{bmatrix} F'(k_t) + N_t & -M_t \\ -N_t & M_t \end{bmatrix} \begin{bmatrix} \mathrm{d}k_t \\ \mathrm{d}C_{t-1} \end{bmatrix}.$$

It is easy to show that $\mathrm{d}k_{t+1} > 0$ and $\mathrm{d}C_t < 0$. Close to $(\overline{k}, \overline{C})$, this matrix has $N \approx 1$. Suppose for one moment that $F'(k)$ was zero, then the matrix has a single nontrivial eigenvalue of $M + N$, which exceeds 1 strictly in the limit, and the associated eigenspace is spanned by $(1, -1)$. The trivial one is spanned by (N, M). In particular, the latter eigenvector is not collinear with the initial perturbation $(0, -1)$, which implies that $\mathrm{d}k_\infty > 0$ and $\mathrm{d}C_\infty < 0$. Hence, $\hat{k}_\infty > k_\infty = \overline{k}$ and $\hat{C}_\infty < C_\infty = \overline{C}$.

In the graph of Z^*, $(\hat{k}_\infty, \hat{C}_\infty)$ locates in the bottom right of $(\overline{k}, \overline{C})$, that means this point is interior. More formally, this means there must exists a time $T' > 0$ for which the continuation value of $(k_{T'+1}, C_{T'})$ is strictly dominated by the one for $(\hat{k}_{T'+1}, \hat{C}_{T'})$, that is, $V(k_{T'+1}, C_{T'}) < V(\hat{k}_{T'+1}, \hat{C}_{T'})$.

Since $T' > T$, we have $V(k_{T+1}, C_T) < V(k_{T+1}, C_T - \varepsilon)$. Thus, V must increase if C_T is lowered, for a path starting at (k_{T+1}, C_T), for large enough T. This proves that the value function is locally decreasing in C at that point. □

Lemma 4.14

An optimal path converges to $(\underline{k}, \underline{C})$.

Proof. Suppose there was an optimal path converging to $(\overline{k}, \overline{C})$. By Lemma 4.13, this means that the value function is locally decreasing around the optimal path (k_{t+1}, C_t), $\forall t > T$, with $T > 0$ large enough.

Consider the following feasible variation for $t = -1, 0, \dots, T$, $\hat{C}_t = C_t - C_t \mathrm{d}\varepsilon_t$, $\hat{k}_{t+1} = k_{t+1}$, and so that $\hat{x}_t = x_t - C_t \mathrm{d}\varepsilon_t$, where the law of motion of $\mathrm{d}\varepsilon_t$ is[5]

$$\mathrm{d}\varepsilon_t = \left(1 - \frac{1}{\sigma}\frac{C_t}{x_t}\right)^{-1} \mathrm{d}\varepsilon_{t-1}.$$

Workers' consumption increases with this variation by $\mathrm{d}c_t = C_t \mathrm{d}\varepsilon_t > 0$. Therefore, the value of this path changes by

$$\mathrm{d}V = \sum_{t=0}^{T} \beta^t u'(c_t)\mathrm{d}c_t + \beta^{T+1}[V(\hat{k}_{T+1}, \hat{C}_{T+1}) - V(k_{T+1}, C_T)].$$

By $\mathrm{d}c_t > 0$, we know that the first term is positive. Then by Lemma 4.13, the second term is positive, too. Hence, $\mathrm{d}V > 0$, which is contradicting the optimality of $\{k_{t+1}, c_t\}$.

Finally, according to Lemma 4.12, an optimal path converges to $(\underline{k}, \underline{C})$. □

5 Inverse Elasticity Formula

5.1 Elasticity with General Saving Function

In this section, we modify the model and assume capitalists behave according to a general (ad hoc) saving rule instead of Euler equation,

$$k_{t+1} = S(R_t k_t; R_{t+1}, R_{t+2}, \dots),$$

where $S(R_t k_t; R_{t+1}, R_{t+2}, \dots) \in [0, I_t]$ is a continuously differentiable function taking as arguments current wealth $I_t = R_t k_t \geq 0$ and future interest rates $\{R_{t+1}, R_{t+2}, \dots\} \in \mathbb{R}_+^{\mathbb{N}}$. Assume that $S_I > 0$.

Let $\gamma = 0$. The planning problem is then

$$\max_{\{c_t, R_t . k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$F(k_t) = c_t + R_t k_t + g,$$
$$k_{t+1} = S(I_t; R_{t+1}, R_{t+2}, \dots),$$

with k_0 given.

Proposition 5.1

Suppose $\gamma = 0$ and assume the savings function is decreasing in future rates, $S_R(I; R_1, R_2, \dots) \leq 0$, $\forall t = 1, 2, \dots$, $\forall \{I, R_1, R_2, \dots\}$. If the optimum converges to an interior steady state in c , k, and R, and at the steady state $\beta R S_I \neq 1$, then the limit tax rate is positive and $\beta R S_I < 1$.

[5] Observe that this equation is precisely the relation which ensures that the variation satisfies all the constraints of the system (RC and ICC). Easy to prove that this expression is well defined.

Proof. At some time $\tau < t$, we denote the derivatives of S by $S_{I,\tau} \equiv \frac{\partial S_\tau}{\partial I_\tau}$ and $S_{\tau,R_t} \equiv \frac{\partial S_\tau}{\partial R_t}$.

Define $\omega_\tau \equiv \frac{\mathrm{d}W_\tau}{\mathrm{d}k_{\tau+1}}$, which corresponds to the response in welfare W_τ, measured in units of period τ utility, of a change in savings by an infinitesimal unit between periods τ and $\tau+1$. According to the planning problem,

$$\omega_\tau = \frac{\mathrm{d}W_\tau}{\mathrm{d}k_{\tau+1}} = \sum_{\tau' \geq \tau+1} \beta^{\tau'-\tau} u'(c_{\tau'}) \frac{\mathrm{d}c_{\tau'}}{\mathrm{d}k_{\tau+1}}.$$

Since we have

$$\frac{\mathrm{d}c_{\tau'}}{\mathrm{d}k_{\tau+1}} = \frac{\mathrm{d}c_{\tau'}}{\mathrm{d}k_{\tau'}} \frac{\mathrm{d}k_{\tau'}}{\mathrm{d}k_{\tau+1}} = \frac{\mathrm{d}c_{\tau'}}{\mathrm{d}k_{\tau'}} \prod_{s=\tau+1}^{\tau'-1} \frac{\mathrm{d}k_{s+1}}{\mathrm{d}k_s} = [F'(k_{\tau'}) - R_{\tau'}] \cdot \prod_{s=\tau+1}^{\tau'-1} S_{I,s} R_s,$$

then

$$\omega_\tau = \sum_{\tau' \geq \tau+1} \beta^{\tau'-\tau} u'(c_{\tau'}) [F'(k_{\tau'}) - R_{\tau'}] \left(\prod_{s=\tau+1}^{\tau'-1} S_{I,s} R_s \right). \tag{5.1}$$

Consider the effect of a one-time change in the capital tax, effectively changing R_t to $R_t + \mathrm{d}R_t$ in period t as a perturbation. Decomposing the effects on total welfare:

$$\begin{aligned}
\mathrm{d}W_t &= \sum_{\tau=0}^{t-1} \frac{\mathrm{d}W_\tau}{\mathrm{d}k_{\tau+1}} \frac{\mathrm{d}k_{\tau+1}}{\mathrm{d}R_t} \mathrm{d}R_t + \frac{\mathrm{d}W_t}{\mathrm{d}k_{t+1}} \frac{\mathrm{d}k_{t+1}}{\mathrm{d}R_t} \mathrm{d}R_t + \frac{\mathrm{d}W_t}{\mathrm{d}c_t} \frac{\mathrm{d}c_t}{\mathrm{d}R_t} \mathrm{d}R_t \\
&= \sum_{\tau=0}^{t-1} \omega_\tau S_{\tau,R_t} \mathrm{d}R_t + \omega_t S_{I,t} k_t \mathrm{d}R_t - u'(c_t) k_t \mathrm{d}R_t.
\end{aligned}$$

Three types of effects on total welfare:

(i) $S_{\tau,R_t} \mathrm{d}R_t$, change in savings in period $\tau < t$;

(ii) $S_{I,t} k_t \mathrm{d}R_t$, change in savings in period t;

(iii) $k_t \mathrm{d}R_t$, change in workers' income in period t.

Along the optimal path, $\mathrm{d}W/\mathrm{d}R = 0$, that is,

$$\omega_t S_{I,t} - u'(c_t) = -\frac{1}{k_t} \sum_{\tau=0}^{t-1} \beta^{\tau-t} \omega_\tau S_{\tau,R_t}. \tag{5.2}$$

By optimization over R_0, we find the condition

$$\omega_0 S_{I,0} k_0 - u'(c_0) k_0 = 0. \tag{5.3}$$

This shows that $S_{I,0} > 0$ and so $\omega_0 \in (0, \infty)$.

Equation (5.1) can be also written as

$$\omega_t = u'(c_{\tau+1})[F'(k_{\tau+1}) - R_{\tau+1}] + \sum_{\tau' \geq \tau+2} \beta^{\tau'-\tau} u'(c_{\tau'}) [F'(k_{\tau'}) - R_{\tau'}] \left(\prod_{s=\tau+1}^{\tau'-1} S_{I,s} R_s \right).$$

We have that

$$\begin{aligned}
&\sum_{\tau' \geq \tau+2} \beta^{\tau'-\tau} u'(c_{\tau'}) [F'(k_{\tau'}) - R_{\tau'}] \left(\prod_{s=\tau+1}^{\tau'-1} S_{I,s} R_s \right) \\
&= \sum_{\tau' \geq (\tau+1)+1} \beta \cdot \beta^{\tau'-(\tau+1)} u'(c_{\tau'}) [F'(k_{\tau'}) - R_{\tau'}] \cdot S_{I,\tau+1} R_{\tau+1} \cdot \left(\prod_{s=(\tau+1)+1}^{\tau'-1} S_{I,s} R_s \right) \\
&= \beta S_{I,\tau+1} R_{\tau+1} \omega_{\tau+1}.
\end{aligned}$$

Thus, the ω_t satisfy the recursion

$$\omega_\tau = u'(c_{\tau+1})[F'(k_{\tau+1}) - R_{\tau+1}] + \beta S_{I,\tau+1} R_{\tau+1} \omega_{\tau+1}.$$

According to this equation, since the steady state is interior, $R_{\tau+1} > 0, \forall \tau$ (o.w. capital would be zero forever), it follows that $\omega_\tau < \infty, \forall \tau$. Combining with the assumption $S_{\tau,R_t} \leq 0$, in equation (5.2) we know that if $\omega_\tau > 0$ $\omega_t S_{I,t} - u'(c_t) \geq 0$.

Using the initial condition (5.3), this proves by induction that

$$\omega_t S_{I,t} - u'(c_t) \geq 0, \quad \forall t > 0. \tag{5.4}$$

Let $\Delta_t \equiv F'(k_t) - R_t$, suppose the economy were converging to an interior steady state with non-positive limit tax, i.e., $\Delta_t \to \Delta \leq 0$, $c_t \to c > 0$, and $S_{I,t} R_t \to S_I R > 0$. If $\Delta < 0$, then ω_t must eventually become negative, a contradiction to (5.3).

Hence, we suppose $\Delta_t \to \Delta = 0$. To discuss the law of motion of ω_t, we have to distinguish the factor $\beta S_I R$ by two cases (note that $\beta S_I R \neq 1$).

Firstly, suppose that $\beta S_I R > 1$. In this case, $\prod_{s=1}^{\tau} (\beta S_{I,s} R_s)$ diverges to ∞. Since w_0 is finite, we have

$$\bar{w}_\tau \equiv \sum_{\tau' \geq \tau+1} \beta u'(c_{\tau'}) \Delta_{\tau'} \prod_{s=1}^{\tau'-1} (\beta S_{I,sR_s}) \to 0, \quad \tau \to \infty,$$

then

$$\omega_\tau = \left(\prod_{s=1}^{\tau} (\beta S_{I,s} R_s) \right)^{-1} \bar{w}_\tau \to 0.$$

In fact, $\lim w_t \geq u'(c)/S_I$ by (5.4), contradicting this result converging to zero. Therefore, this case is not compatible with any interior steady state.

In this case, we do not use the fact that $\Delta_t \to 0$. Even if $\Delta_t > 0$ (positive limit tax), we still have this conclusion. This argument preclude $\beta S_I R > 1$ in all cases.

Respectively, suppose that $\beta S_I R < 1$. $\forall \varepsilon > 0$, $\exists \bar{\tau}$ s.t. $\beta S_{I,s} R_s < b$ for some $b < 1$, $\forall \tau > \bar{\tau}$, $|u'(c_{\tau'}) \Delta_{\tau'}| < \varepsilon(1-b)$. Then,

$$|\omega_\tau| \leq \sum_{\tau'=\tau+1}^{\infty} \varepsilon(1-b) b^{\tau'-1-\tau} = \varepsilon.$$

This is towards a contradiction again.

In sum, the capital tax $\mathcal{T}_t = \Delta_t / F'(k_t)$ must converge to a positive number at the interior steady state. □

Now we begin to derive the inverse elasticity rule with the optimal tax rate.

Proposition 5.2

Assume that

$$\sum_{\tau=1}^{t} \beta^{-\tau} \frac{\omega_{t-\tau} k_{t-\tau}}{\omega_t k_t} \varepsilon_{S_{t-\tau}, R_t} \to \sum_{\tau=1}^{\infty} \beta^{-\tau} \varepsilon_{S,\tau} \in [-\infty, +\infty], \quad t \to \infty, \tag{5.5}$$

where

$$\varepsilon_{S,t} := \frac{\partial S/S}{\partial R_t / R_t} (R_0 k_0; R_1, R_2, \dots).$$

Then we have the following inverse elasticity rule,

$$\mathcal{T} = 1 - \frac{R}{R^*} = \frac{1 - \beta R S_I}{1 + \sum_{t=1}^{\infty} \beta^{-t+1} \varepsilon_{S,t}}. \tag{5.6}$$

Proof. Because $\beta S_I R < 1$ by Proposition 5.1, we can take the limit of equation (5.1):

$$\omega = \lim_{t\to\infty} \sum_{\tau' \geq \tau+1} \beta^{\tau'-\tau} \left(\prod_{s=\tau+1}^{\tau'-1} S_{I,s} R_s \right) \cdot \left[u'(c_{\tau'})[F'(k_{\tau'}) - R_{\tau'}] \right] = \frac{\beta}{1-\beta S_I R} u'(c)[F'(k) - R].$$

Rearrange equation (5.2) into a form with elasticity. Dividing by ω_t one yields

$$S_{I,t} - \frac{u'(c_t)}{\omega_t} = -\frac{1}{\omega_t k_t} \sum_{\tau=0}^{t-1} \beta^{\tau-t} \omega_\tau S_{\tau,R_t}.$$

We have

$$\begin{aligned}
\sum_{\tau=0}^{t-1} \beta^{\tau-t} \omega_\tau S_{\tau,R_t} &= \sum_{\tau=1}^{t} \beta^{-\tau} \omega_{t-\tau} S_{t-\tau,R_t} \\
&= \sum_{\tau=1}^{t} \beta^{-\tau} \omega_{t-\tau} \varepsilon_{t-\tau,R_t} \cdot \frac{S_{t-\tau}}{R_t} \\
&= \sum_{\tau=1}^{t} \beta^{-\tau} \omega_{t-\tau} k_{t-\tau} \varepsilon_{t-\tau,R_t}.
\end{aligned}$$

Then,

$$S_{I,t} - \frac{u'(c_t)}{\omega_t} = -\sum_{\tau=1}^{t} \beta^{-\tau} \frac{\omega_{t-\tau} k_{t-\tau}}{w_t k_t} \varepsilon_{S_{t-\tau},R_t}. \tag{5.7}$$

Under the assumption of (5.5)[6] , we can characterize the limit of equation (5.7) as $t \to \infty$,

$$S_{I,t} - \frac{u'(c)}{\omega} = -\sum_{\tau=1}^{\infty} \beta^{-\tau} \varepsilon_{S,\tau}. \tag{5.8}$$

Now we need to distinguish two cases according to whether $\omega = 0$ or $\omega \neq 0$.

First, the trivial one, if $\omega = 0$ (or $\mathcal{T} = 0$), then the RHS of equation (5.8) is infinity (plus or minus). Therefore, the inverse formula holds in this case as both sides of (5.6) converge to zero.

Second, if $\omega \neq 0$, using the definition of ω, (5.8) can be written as the form about $\beta S_I R$,

$$\frac{\beta S_I R}{1-\beta S_I R}[F'(k) - R] - R = -\frac{1}{1-\beta S_I R}[F'(k) - R] \sum_{\tau=1}^{\infty} \beta^{-\tau+1} \varepsilon_{S,\tau}.$$

---

[6] This is actually a regularization assumption (5.5). We discuss its mathematical meaning as below. Define

$$f_t(\tau) := \begin{cases} \beta^{-\tau} \frac{\omega_{t-\tau} k_{t-\tau}}{\omega_t k_t} \varepsilon_{S_{t-\tau},R_t} & \tau = 1, 2, \ldots, t \\ 0 & \text{o.w.} \end{cases}.$$

At steady state, we have $\frac{\omega_{t-\tau} k_{t-\tau}}{\omega_t k_t} \to 1$ and $\varepsilon_{S_{t-\tau},R_t} \to \varepsilon_{S,\tau}$, $\forall \tau$, so for each fixed τ, $\lim f_t(\tau) = f(\tau)$. Hence, define the limit function

$$f(\tau) = \beta^{-\tau} \varepsilon_{S,\tau},$$

then this assumption is equal to

$$\lim_{t\to\infty} \sum_{\tau=1}^{\infty} f_t(\tau) = \sum_{\tau=1}^{\infty} \lim_{t\to\infty} f_t(\tau) \equiv \sum_{\tau=1}^{\infty} f(\tau).$$

By Lebesgue's Dominated Convergence Theorem, this requires $\exists g(\tau)$, $\sum_{\tau=1}^{\infty} g(\tau) < \infty$, s.t. $\forall \varepsilon > 0$, $\exists T$, $\forall t > T$,

$$|f_t(\tau) - f(\tau)| \leq g(\tau) \cdot \varepsilon, \quad \tau = 1, 2, \ldots.$$

Notice that $F'(k) - R = \frac{\mathcal{T}}{1-\mathcal{T}}R$. Therefore, by dividing $[F'(k) - R]$ on both sides, we can rearrange this condition to

$$\frac{\beta S_I R}{1-\beta S_I R} - \frac{1-\mathcal{T}}{\mathcal{T}} = -\frac{1}{1-\beta S_I R}\sum_{\tau=1}^{\infty}\beta^{-\tau+1}\varepsilon_{S,\tau}.$$

Thus,

$$\mathcal{T} = \frac{1-\beta S_I R}{1+\sum_{t=1}^{\infty}\beta^{-t+1}\varepsilon_{S,t}}.$$

□

The following corollary is against the conclusion of Piketty and Saez (2013).

Corollary 5.3

Under the conditions of Proposition 5.1, the inverse elasticity formula (5.6) cannot hold if

$$1+\sum_{t=1}^{\infty}\beta^{-t+1}\varepsilon_{S,t} < 0.$$

Proof. Conditional on convergence to an interior steady state, we have proved that the limit tax rate is positive. If now the inverse elasticity formula implies a negative tax rate, then either the regularity condition (5.5) for the inverse elasticity rule is not satisfied or the allocation does not converge to an interior steady state. □