On the Mechanics of Economic Development

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March 27th, 2025

Prelims

- Discounting in continuous time.
- Some useful functions:
    - CES and Cobb-Douglas
    - CRRA
- Current value Hamiltonian:
    - FOC
    - TVC
- Some properties on BGP and competitive equilibrium.

Outline

The Malthusian Trap

- Increased food production as a result of advanced agricultural techniques creates higher population levels.
- These higher population levels then lead to food shortages, as the new population must live on land that was previously used for crops.
- Even though technological advancement would typically lead to income gains per capita, the gains wouldn't be achieved because, in practice, the advancement creates population growth[1].

[1] Malthus, T. (1798). An essay on the principle of population.

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- Solow and Swan: exogenous technical change.
  - Cannot assure growth in per capita income under an exogenously given and constant technology.
- Lucas: accumulation in human capital.
  - Can also explain the "demographic transition"—rising longevity and declining fertility.
- Romer: endogenous technical change.
  - Human capital is exogenous, so there is never any demographic transition.

Assumptions

- Closed economy with competitive markets
- Identical, rational agents
- Infinitely-lived households
- CRS technology

Setup: Utility

- $N(t)$, the persons devoted to production. (i.e. the labor)
  - $\lambda = \frac{\dot{N}(t)}{N(t)} > 0$, the growth rate, is exogenously given.
- $\rho > 0$, the discount rate.
- $\sigma > 0$, the reciprocal of the intertemporal elasticity of substitution.
- $c(t)$, real, per-capita consumption of units of a single good.
  - $\kappa = \frac{\dot{c}(t)}{c(t)}$, the growth rate.
- CRRA utility function:

$$U = \frac{c(t)^{1-\sigma} - 1}{1 - \sigma}$$

Setup: Output

- $K(t)$, the total stock of capital.
- $A(t)$, the technology.
  - $\mu = \frac{\dot{A}(t)}{A(t)} > 0$, the growth rate, is exogenously given.
- $\beta \in (0, 1)$, the capital's share.
- Cobb-Douglas production function:

$$Y = A(t)K(t)^{\beta}N(t)^{1-\beta}$$

- The output is equal to the total consumption plus the net investment.

$$Y = N(t)c(t) + \dot{K}(t)$$

Optimization

- UMP:

$$\max_{c(t)} \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \cdot N(t)\mathrm{d}t$$

$$\text{s.t.} \quad N(t)c(t) + \dot{K}(t) = A(t)K(t)^\beta N(t)^{1-\beta}$$

- The current value Hamiltonian:

$$\mathcal{H} = N \cdot \frac{c^{1-\sigma} - 1}{1-\sigma} + \theta \cdot (AK^\beta N^{1-\beta} - Nc)$$

Optimization

- FOC:

$$\frac{\partial \mathcal{H}}{\partial c} = 0$$
$$\frac{\partial \mathcal{H}}{\partial K} = \rho\theta(t) - \dot{\theta}(t)$$

- TVC:

$$\lim_{t \to \infty} \mathrm{e}^{-\rho t}\theta(t)K(t) = 0$$

Results

- The marginal productivity of capital condition:

$$\beta A(t) N(t)^{1-\beta} K(t)^{\beta-1} = \rho + \sigma\kappa$$

- On the BGP, the growth rate of capital is constant. With some math, we obtain

$$\frac{\dot{K}(t)}{K(t)} = \kappa + \lambda$$

- Thus, the growth rate of per-capita consumption must be constant. Actually we have

$$\kappa = \frac{\mu}{1-\beta}$$

- The constant, balanced net saving rate s is defined by

$$s = \frac{\dot{K}(t)}{N(t)c(t) + \dot{K}(t)} = \frac{\beta(\kappa+\lambda)}{\rho + \sigma\kappa}$$

Paradox

- Growth effect: changes in parameters that alter growth rates along balanced paths.
- Level effect: changes that raise or lower balanced growth paths without affecting their slope.
- Solow's conclusion: changes in savings rates are level effects[2].

[2] Solow, R. M. (1956). A contribution to the theory of economic growth. *The Quarterly Journal of Economics*, 70(1), 65-94.

Paradox

- Krueger (1983) and Harberger (1984) both claimed that identify inefficient barriers to trade as a limitation on growth, and their removal as growth effect.
- But by the neoclassical model, it is a level effect, analogous to the one-time shifting upward in production possibilities.

Motivation

- In Solow model, (exogenous) technological change is the only driver of economic growth ($\kappa = \frac{\mu}{1-\beta}$).
- The necessary condition of convergence: capital has diminishing marginal returns. ($\beta < 1$)
    - See Romer (1986)[3].

[3] Romer, P. M. (1986). Increasing returns and long-run growth. *Journal of Political Economy*, 94(5), 1002-1037.

Motivation

- Considering an alternative, or at least a complementary, engine of growth to the technological change that serves this purpose in the Solow model, retaining for the moment the other features of that model (in particular, its closed character).
- Adding what Schultz (1963) and Becker (1964) call 'human capital' to the model, doing so in a way that is very close technically lo similarly motivated models of Arrow (1962), Uzawa (1965) and Romer (1986).

Motivation

- What is human capital?
- An intangible asset, best thought of as a stock of embodied and disembodied knowledge, comprising education, information, health, entrepreneurship, and productive and innovative skills, that is formed through investments in schooling, job training, and health, as well as through research and development projects and informal knowledge transfers[4].

[4] Ehrlich, I., & Murphy, K. M. (2007). Why does human capital need a journal?. *Journal of Human Capital*, 1(1), 1-7.

Uzawa's Setup: Labor

- $N(t)$, the total labor force.
- The growth rate λ is constant.
- $N_P(t)$, labor employed in material production.
- $N_E(t)$, labor employed in the educational sector.
- $\eta = \frac{N_E(t)}{N(t)}$, the proportion of labor employed in the educational sector.
- $N_P(t) + N_E(t) \leq N(t)$
- $h(t)$, the labor efficiency (i.e. human capital).

Uzawa's Setup: Human Capital

- $h(t)$, the labor efficiency (i.e. human capital).
- The accumulation of human capital:

$$\dot{h}(t) = h(t)\delta(\eta)$$

  where $\delta(\eta)$ is increasing and concave for $0 \leq \eta \leq 1$.
- The output function:

$$Y = A[K(t), h(t)L_p(t)]$$

Uzawa's Setup: Physical Capital

- $C(t)$, aggregate consumption.
- $I(t)$, aggregate investment.
- Then we have:

$$Y = C(t) + I(t)$$

- The accumulation of physical capital:

$$\dot{K}(t) = K(t)[I(t) - \phi K(t)]$$

where ϕ stands for the rate of depreciation of physical capital.

Lucas' Contribution

- Consider not only the internal effect but also the external effect, which causes the endogenous growth.
- Expand a new mechanism of the accumulation of human capital–learning by doing.

Setup: Human Capital

- N workers in total, with skill levels $h \in (0, +\infty)$.
  - $N(h)$ workers with skill level h, $N = \int_0^\infty N(h)\mathrm{d}h$.
- A worker with skill h devotes the fraction $u(h)$ of his non-leisure time to current production, and the remaining $1 - u(h)$ to human capital accumulation.
- Then, the effective workforce in production $N^e = \int_0^\infty u(h)N(h)h\mathrm{d}h$.
- The average level of skill (human capital) is defined by

$$\bar{h} = \frac{\int_0^\infty hN(h)\mathrm{d}h}{\int_0^\infty N(h)\mathrm{d}h}$$

- If all workers are identical, correspondingly, $N^e = uhN$, $\bar{h} = h$.

Setup: Output

- The human capital accumulation has two different effect on promoting economic growth:
  - Internal effect: increasing the marginal product of labor by N^e.
  - External effect: increasing the output directly by $\bar{h}$. (as the technology)
- The output function can be written as

$$
\begin{aligned}
Y &= AK(t)^{\beta}[N^e(t)]^{1-\beta}\bar{h}(t)^{\gamma} \\
&= AK(t)^{\beta}[u(t)h(t)N(t)]^{1-\beta}\bar{h}(t)^{\gamma}
\end{aligned}
$$

where γ is the power of the external effect.

Setup: Accumulation of Human Capital

- We postulate a technology relating the growth of human capital:

$$\dot{h}(t) = h(t)^{\xi} G[1 - u(t)]$$

  where G is increasing, with $G(0) = 0$.

- Following Uzawa(1965) and Rosen(1976), we assume $\xi = 1$ and G is linear:

$$\dot{h}(t) = h(t)\delta[1 - u(t)]$$

  where the maximal growth rate $\delta > 0$.

- The growth rate of human capital $\nu = \frac{\dot{h}(t)}{h(t)} = \delta(1 - u)$ on BGP.

Divergence

- In the presence of the external effect, it will not be the case that optimal growth paths and competitive equilibrium paths coincide.
  - eg. academic tenure
- For the central decision maker, since all workers are identical, $\forall t$, $\bar{h}(t) = h(t)$.
- For the household decision maker, given $\bar{h}(t)$, consider the problem the private sector, consisting of atomistic households and firms, would solve if each agent expected the average level of human capital to follow the path $\bar{h}(t)$.(i.e. taking $\bar{h}(t)$ as exogenously determined).

Optimization: Efficient Path

- UMP of the central decision maker:

$$\max_{c(t),u(t)} \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{c(t)^{1-\sigma}-1}{1-\sigma} \cdot N(t)\mathrm{d}t$$

$$\text{s.t.} \quad N(t)c(t) + \dot{K}(t) = AK(t)^\beta [u(t)h(t)N(t)]^{1-\beta} \bar{h}(t)^\gamma$$

$$\dot{h}(t) = h(t)\delta[1-u(t)]$$

$$\bar{h}(t) = h(t)$$

- The current value Hamiltonian:

$$\mathcal{H} = N \cdot \frac{c^{1-\sigma}-1}{1-\sigma} + \theta_1 \cdot [AK^\beta (uNh)^{1-\beta} h^\gamma - Nc] + \theta_2 \cdot [\delta h(1-u)]$$

Optimization: Efficient Path

- FOC:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial c}: &\quad c^{-\sigma} = \theta_1 \\
\frac{\partial \mathcal{H}}{\partial u}: &\quad \theta_1(1-\beta)AK^{\beta}(uNh)^{-\beta}Nh^{1+\gamma} = \theta_2\delta h \\
\frac{\partial \mathcal{H}}{\partial K}: &\quad \dot{\theta}_1 = \rho\theta_1 - \theta_1\beta AK^{\beta-1}(uNh)^{1-\beta}h^{\gamma} \\
\frac{\partial \mathcal{H}}{\partial h}: &\quad \dot{\theta}_2 = \rho\theta_2 - \theta_1(1-\beta+\gamma)AK^{\beta}(uN)^{1-\beta}h^{1-\beta+\gamma} - \theta_2\delta(1-u)
\end{aligned}
$$

Optimization: Equilibrium Path

- UMP of the household decision maker:

$$\max_{c(t),u(t)} \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{c(t)^{1-\sigma}-1}{1-\sigma} \cdot N(t)\mathrm{d}t$$

$$\text{s.t.} \quad N(t)c(t) + \dot{K}(t) = AK(t)^\beta [u(t)h(t)N(t)]^{1-\beta} \bar{h}(t)^\gamma$$

$$\dot{h}(t) = h(t)\delta[1-u(t)]$$

- The current value Hamiltonian:

$$\mathcal{H} = N \cdot \frac{c^{1-\sigma}-1}{1-\sigma} + \theta_1 \cdot [AK^\beta (uNh)^{1-\beta} \bar{h}^\gamma - Nc] + \theta_2 \cdot [\delta h(1-u)]$$

Optimization: Equilibrium Path

- FOC:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial c}: &\quad c^{-\sigma} = \theta_1 \\
\frac{\partial \mathcal{H}}{\partial u}: &\quad \theta_1(1-\beta)AK^{\beta}u^{-\beta}(Nh)^{1-\beta}\bar{h}^{\gamma} = \theta_2 \delta h \\
\frac{\partial \mathcal{H}}{\partial K}: &\quad \dot{\theta}_1 = \rho\theta_1 - \theta_1 \beta A K^{\beta-1}(uNh)^{1-\beta}\bar{h}^{\gamma} \\
\frac{\partial \mathcal{H}}{\partial h}: &\quad \dot{\theta}_2 = \rho\theta_2 - \theta_1(1-\beta)AK^{\beta}(uNh)^{1-\beta}\bar{h}^{\gamma} - \theta_2\delta(1-u)
\end{aligned}$$

- Since the market clearing, $\bar{h}(t) = h(t)$, the forth FOC can be written as

$$\dot{\theta}_2 = \rho\theta_2 - \theta_1(1-\beta)AK^{\beta}(uN)^{1-\beta}h^{1-\beta+\gamma} - \theta_2\delta(1-u)$$

Divergence

- If $\gamma = 0$, two paths are the same. It is the presence of the external effect $\gamma > 0$ that creates a divergence between the 'social' valuation and the private valuation.

Results

- The marginal productivity of capital condition:

$$\beta A K(t)^{\beta-1}[u(t)N(t)h(t)]^{1-\beta}h(t)^{\gamma} = \rho + \sigma\kappa$$

- It is easy to verify that $\frac{\dot{K}(t)}{K(t)} = \kappa + \lambda$ in this model.
- Solve the relationship of the two growth rates between the physical capital and the human capital, we obtain

$$\kappa = \left(\frac{1-\beta+\gamma}{1-\beta}\right)\nu$$

Results

- Then we turn to the determinants of ν.
- The efficient growth rate is:

$$\nu_1^* = \sigma^{-1}\left[\delta - \frac{1-\beta+\gamma}{1-\beta}(\rho-\lambda)\right]$$

- And the equilibrium growth rate is:

$$\nu_2^* = [\sigma(1-\beta+\gamma)-\gamma]^{-1}[(1-\beta)(\delta-(\rho-\lambda))]$$

Restriction

- The rates ν_1^* and ν_2^* must not exceed the maximum feasible rate δ. This restriction can be seen to require

$$\sigma \geq 1 - \frac{1-\beta}{1-\beta+\gamma}\frac{\rho-\lambda}{\delta}$$

so the model cannot apply at the intertemporal substitutability of consumption that is too high.

Efficiency

- For the case $\sigma = 1$, the inefficient part can be written as

$$\nu_1^* - \nu_2^* = \frac{\gamma}{1-\beta+\gamma}(\rho - \lambda)$$

- Thus the inefficiency is small when either the external effect is small ($\gamma \to 0$) or the discount rate is low ($\rho - \lambda \to 0$).

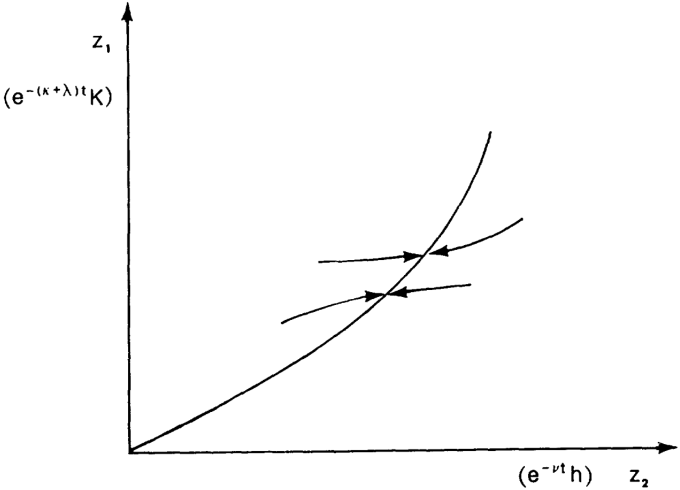
Dynamics

- We define two normalized variables $z_1(t) = \mathrm{e}^{-(\kappa+\lambda)t}K(t)$ and $z_2(t) = \mathrm{e}^{-\nu t}h(t)$. Inserting these variables into the formula which describes the marginal productivity of capital condition, we obtain

$$[\beta A N(0)^{1-\beta} u^{1-\beta}] z_1^{\beta-1} z_2^{1-\beta+\gamma} = \rho + \sigma\kappa$$

- It is a fact that all pairs (z_1, z_2) satisfying the formula above correspond to balanced paths.

Dynamics



Dynamics

- Along this curve, then, returns to capital are constant and also constant over time even though capital stocks of both kinds are growing.
- When $\gamma = 0$, it will also be true that the real wage rate for labor of a given skill level (the marginal product of labor) is constant along the curve.
- When $\gamma > 0$, the real wage increases as one moves up the curve.

Wages

- Along this curve, for wages $w = \frac{\partial Y}{\partial N}$, we have the elasticity formula

$$\frac{\partial w/w}{\partial K/K} = \frac{(1+\beta)\gamma}{1-\beta+\gamma}$$

  so that wealthier countries have higher wages than poorer ones for labor of any given skill.
- In all countries, wages at each skill level grow at the rate $\omega = \frac{\gamma}{1-\beta}\nu$, then taking skill growth into account as well, wages grow at

$$\omega + \nu = \frac{1-\beta+\gamma}{1-\beta}\nu = \kappa$$

  or at a rate equal to the growth rate in the per-capita stock of physical capital.

Assumptions

- Closed economy again
- No physical capital
- Constant population

Setup: Production

- Two consumption goods, c_1 and c_2.
- h_i, human capital specialized to the production of good i.
- u_i, the fraction of the workforce devoted to producing good i.
  - $u_i \geq 0$, $u_1 + u_2 = 1$.
- Production function with Ricardian technology:

$$c_i(t) = h_i(t)u_i(t)N(t)$$

Setup: Accumulation of Human Capital

- Through learning-by-doing:

$$\dot{h}_i(t) = h_i(t)\delta_i u_i(t)$$

- Assume that $\theta_1 > \theta_2$, so that good 1 is taken to be the 'high-technology' good.
- The effects of $h_i(t)$ are entirely external.

Setup: Utility

- The individual consumer has NO intertemporal decision.
  - No physical capital accumulation.
  - Purely external human capital accumulation.
- CES utility function[5]:

$$U(c_1, c_2) = (\alpha_1 c_1^{-\rho} + \alpha_2 c_2^{-\rho})^{-\frac{1}{\rho}}$$

where $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 = 1$, $\rho > -1$, and $\sigma = \frac{1}{1+\rho}$ is the elasticity of substitution between c_1 and c_2.

[5] NOTICE: the parameters ρ and σ represent **completely different** aspects of preferences in this section from those they represented in the previous sections.

Price and Allocation

- Take the first good as numeraire, and let $(1, q)$ be the equilibrium prices. Then q must equal the MRS in consumption:

$$q = \frac{U_2(c_1, c_2)}{U_1(c_1, c_2)} = \frac{\alpha_2}{\alpha_1}\left(\frac{c_2}{c_1}\right)^{-(1+\rho)}$$

- The relative prices are dictated by the human capital endowments $q = \frac{h_1}{h_2}$.
- Solve the equilibrium workforce allocation:

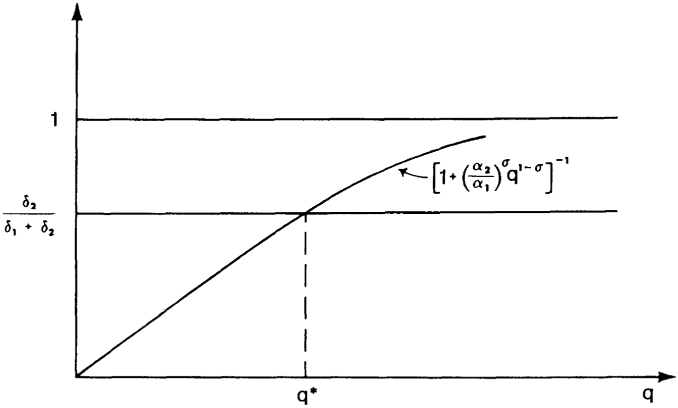
$$\frac{u_2}{u_1} = \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma}\left(\frac{h_2}{h_1}\right)^{\sigma-1}$$

Dynamics

- Solving first for the autarchy price path:

$$\frac{\dot{q}}{q} = (\delta_1 + \delta_2)\left[1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}\right]^{-1} - \delta_2$$

- When $\sigma > 1$, the function $\left[1 + \left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}\right]^{-1}$ is increasing. To the left of q^*, $\dot{q} < 0$, $q(t)$ tends to 0. To the right, $\dot{q} > 0$, $q(t)$ grows without bound.
- Thus the system in autarchy converges to specialization in one of the two goods. (unless $q(0) = q^*$)


1
$\frac{\delta_2}{\delta_1 + \delta_2}$
$\left[1+\left(\frac{\alpha_2}{\alpha_1}\right)^{\sigma} q^{1-\sigma}\right]^{-1}$
q^*
q

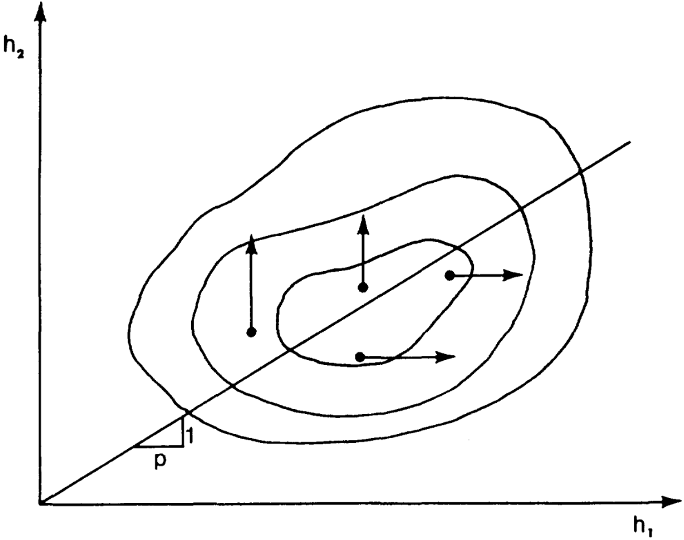
Dynamics

- When $\sigma < 1$, q^* becomes a stable stationary point. At this point, the workforce is so allocated as to equate $\delta_1 u_1$ and $\delta_2 u_2$.
- When $\sigma = 1$, the workforce is initially allocated as dictated by the demand weights, $u_i = \alpha_i$, and this allocation is maintained forever. The autarchy price grows (or shrinks) at the constant rate $\alpha_1\delta_1 - \alpha_2\delta_2$ forever.

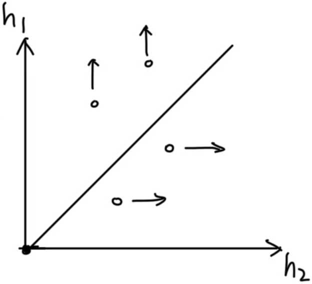
International Trade

- Perfectly free trade in the two final goods.
- With a continuum of small countries.
- Prices in all countries will equal world prices $(1, p)$, say, and each country will take p as given.
- Each country has a endowment distribution, (h_1, h_2) indicates the comparative advantage.

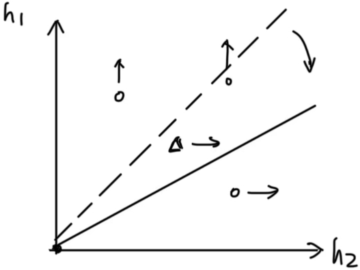
Dynamics



Dynamics



State I



State II

Dynamics

- Is it possible that these price movements will induce any country to switch its specialization from one good to the other?
- The inequality that rules this possibility out is

$$\sigma \geq 1 - \frac{\delta_2}{\delta_1}$$

- Adopting the case $\sigma > 1$, this inequality is consistently satisfied.
- With no producer switching, the growth rate of p can be shown as:

$$\frac{\dot{p}}{p} = \frac{\delta_1 - \delta_2}{\sigma}$$

Cross-National Differences

- Output of the good 1 producers grows at the rate δ_1. Output of the good 2 producers grows at the rate $\delta_2 + \frac{\dot{q}}{q} = \delta_2 + \frac{\delta_1 - \delta_2}{\sigma}$.
- In general, countries in equilibrium will undergo constant but not equal growth rates of real output.
- When $\sigma > 1$, $\delta_1 > \delta_2 + \frac{\delta_1 - \delta_2}{\sigma}$, that is, producing (having a comparative advantage in) high-learning goods will lead to higher-than-average real growth only if the two goods are good substitutes.
- Exceptions: The terms-of-trade effects of technological change dominated the direct effects on productivity (which would be the case if $\sigma < 1$), those countries with rapid technological change would enjoy the slowest real income growth.

Comment

- Back to the technology relating the growth of human capital:

$$\dot{h}(t) = h(t)^{\xi} G[1 - u(t)]$$

- If we take $\xi < 1$ in this formulation, so that there is diminishing returns to the accumulation of human capital. It is easy to see the growth rate ν must eventually tend to zero.
- How about $\xi > 1$? See Romer (1990)[6].

[6] Romer, P. M. (1990). Endogenous technological change. *Journal of Political Economy*, 98(5, Part 2), S71-S102.

Comment

- If each individual acquired human capital but if none of this capital were passed on to younger generations, the households' stock would (with a fixed demography) stay constant.
- To obtain the formula in the last page, for a family, one needs to assume both that each individual's capital follows this equation and that the initial level each new member begins with is proportional to.
- Human capital with overlapping generations, see Errhich and Lui (1991)[7].

[7] Ehrlich, I., & Lui, F. T. (1991). Intergenerational trade, longevity, and economic growth. *Journal of Political Economy*, 99(5), 1029-1059.

Comment

- Mankiw, Phelps and Romer (1995)[8] points out that since lifetimes are finite, there is a maximum limit to the amount of human capital that an individual can accumulate. Thus, while increasing human capital (with more years of schooling, for example) may be able to extend the length of the transition period in a growth model, human capital accumulation cannot be the source of perpetual growth.

[8] Mankiw, N. G., Phelps, E. S., & Romer, P. M. (1995). The growth of nations. *Brookings papers on economic activity*, (1), 275-326.

Comment

- Another critical thinking to the internal effect of Uzawa-Lucas model: see Grossman, Helpman, Oberfield and Sampson (2017)[9].
- Balanced growth is possible if schooling is endogenous and capital is more complementary with schooling than with raw labor.

[9] Grossman, G. M., Helpman, E., Oberfield, E., & Sampson, T. (2017). Balanced growth despite Uzawa. *American Economic Review*, 107(4), 1293-1312.

Knowledge is the only instrument of production that is not subject to diminishing returns.
(John Maurice Clark, 1923)

Questions are welcome.