Reading Notes of Auclert, Cai, Rognlie, and Straub (wp) "Optimal Long-Run Fiscal Policy with HA"

George Chou

June 27, 2026

1 Environment

Our economy is a standard heterogeneous-agent economy à la Aiyagari (1995), with flexible labor supply and a general utility function. We abstract away from capital at first to ease exposition. Time t is discrete, and there is no aggregate risk. All agents have perfect foresight w.r.t. aggregate variables.

1.1 Basic Setup

There is a continuum of households, labeled by $i \in [0,1]$. At the beginning of each period, household i draws idiosyncratic productivity $e_{i,t}$ from a positive and recurrent finite-state Markov chain; $e_{i,t}$ is i.i.d. across households, with a mean normalized to 1. We further allow for an end-of-period productivity shock $\varepsilon_{i,t} > 0$ that is i.i.d. over time and across households, and drawn from a smooth distribution with density $\theta(\varepsilon)$ and mean 1[1]. $\varepsilon_{i,t}$ is realized after consumption and labor supply are chosen.

Household i's date-0 expected utility is given by

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t}) \right], \tag{1.1}$$

where $\beta \in (0,1)$ is the private discount factor, $c_{i,t}$ denotes consumption and $n_{i,t}$ the labor supply. $u(c,n)$ is a general per-period utility function satisfying the usual Inada conditions.

Households are able to self-insure against their idiosyncratic productivity shocks by holding assets $a_{i,t}$, earning an interest rate r_t between periods $t-1$ and t. Their budget constraint is given by

$$c_{i,t} + a_{i,t} = (1 + r_t) a_{i,t-1} + w_t e_{i,t} \varepsilon_{i,t} n_{i,t}, \tag{1.2}$$

where $w_t > 0$ is the date-t after-tax wage per effective unit of labor.

Households are subject to a standard borrowing constraint[2]

$$a_{i,t} \geq 0. \tag{1.3}$$

Taken together, households choose $\{c_{i,t}, n_{i,t}, a_{i,t}\}$ in order to maximize (1.1) s.t. (1.2) and (1.3).

[1] We allow for $\varepsilon_{i,t}$ to just ensure that wealth distribution predicted by the model is smooth, which will help with theoretical results. In all numerical exercises, $\varepsilon_{i,t}$ is set to 1.

[2] We also allow for an upper bound on assets $a_{i,t} \leq \bar{a}$ where $\bar{a} \in (0, \infty]$. For theoretical results, we assume $\bar{a} < \infty$ is some arbitrarily large positive constant. For numerical results, we set $\bar{a}$ sufficiently high that it is not binding.

Our main focus is on preferences that are compatible with balanced growth, à la King et al. (1988) (KPR), given by

$$u(c,n) = \frac{\left[c\mathrm{e}^{-v(n)}\right]^{1-\sigma} - 1}{1-\sigma}, \tag{1.4}$$

for some $\sigma > 0$ and $v(n) \geq 0$ denoting the disutility from labor supply.

We focus in particular on the $\sigma = 1$ case, which delivers the standard log-separable utility function,

$$u(c,n) = \ln c - v(n). \tag{1.5}$$

Production is perfectly competitive and linear in effective labor. The unique output good at date t is produced using the technology

$$Y_t = N_t, \tag{1.6}$$

where N_t is the aggregate effective labor supply. This implies that the pre-tax wage $w_t^* = 1$, $\forall t$.

The government is financing an exogenous amount of government spending $G > 0$ using a mix of proportional labor income taxes $\{\tau_t\}$ and government debt $\{B_t\}$. After-tax wages are therefore simply

$$w_t = 1 - \tau_t, \tag{1.7}$$

and tax revenue from labor income taxation is given by $\tau_t N_t$.

The government budget constraint is

$$G + (1 + r_t)B_{t-1} = B_t + \tau_t N_t. \tag{1.8}$$

1.2 Sequence-Space Functions

From a partial equilibrium view, aggregate household behavior in this economy can be expressed entirely as a function of the sequences of interest rates and wages, $\{r_t, w_t\}_{t=0}^{\infty}$. For simplicity, we omit the superscript and subscript hereafter.

Conditional on $\{r_t, w_t\}$, households can solve for their consumption policies $c_t^*(e, a_-)$ and labor supply polices $n_t^*(e, a_-)$. Then the individual saving $\tilde{a}_t^*$ can be written as

$$\tilde{a}_t^*(e, a_-, \varepsilon) = (1 + r_t)a_- + w_t e \varepsilon n_t^*(e, a_-) - c_t^*(e, a_-).$$

We construct a_t^* as the average saving of households (as well as the saving policies), that is,

$$a_t^*(e, a_-) = \int \tilde{a}_t^*(e, a_-, \varepsilon)\theta(\varepsilon)\mathrm{d}\varepsilon.$$

These policies then perfectly determine the evolution of the wealth distribution $\Psi_t(e, a_-)$, starting from some arbitrary given initial distribution $\Psi_0(e, a_-)$. Thus, date-t aggregate household behavior can be represented as

$$A_t = \mathscr{A}_t(\{r_s, w_s\}) \equiv \int a_t^*(e, a_-)\mathrm{d}\Psi_t(e, a_-), \tag{1.9}$$

$$N_t = \mathscr{N}_t(\{r_s, w_s\}) \equiv \int n_t^*(e, a_-)\mathrm{d}\Psi_t(e, a_-), \tag{1.10}$$

$$C_t = \mathscr{C}_t(\{r_s, w_s\}) \equiv \int c_t^*(e, a_-)\mathrm{d}\Psi_t(e, a_-). \tag{1.11}$$

Finally, utilitarian flow utility can be written as

$$U_t = \mathscr{U}_t(\{r_s, w_s\}) \equiv \int u[c_t^*(e, a_-), n_t^*(e, a_-)]\mathrm{d}\Psi_t(e, a_-). \tag{1.12}$$

We make the following assumption to ensure the stationary in the economy, which is standard in all heterogeneous-agent models.

Assumption 1.1

$\forall r < \beta^{-1} - 1$, $\forall w > 0$, $\exists!$ globally stable steady state wealth distribution $\Psi(e, a_-)$.

With stationary, the derivatives of each functions $\mathscr{X} \in \{\mathscr{A}, \mathscr{N}, \mathscr{C}, \mathscr{U}\}$ have specific shapes, namely that of β-quasi-Toeplitz matrices.

Definition 1.2: β-Quasi-Toeplitz Matrix

An infinite matrix $\mathbf{M} = [M_{t,s}] \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is a β-quasi-Toeplitz matrix with symbol vector $\mathbf{a} = \{a_t\} \in \mathbb{R}^{\mathbb{Z}}$ if:

1. $\forall t, s \geq 0$,
$$\lim_{u\to\infty} M_{t+u,s+u} = a_{t-s}. \tag{1.13}$$

2. $\exists \psi \in (0,1)$ s.t.
$$|M_{t,s}| \leq \begin{cases} D\psi^{t-s} & t-s \geq 0 \\ D(\beta\psi)^{-(t-s)} & t-s < 0 \end{cases}, \tag{1.14}$$
where $D > 0$ is a constant.

We describe the asymptotical properties of sequence-space functions $\mathscr{X}$ as below,

Proposition 1.3

Each one of the sequence-space functions $\mathscr{X}$ is a well-defined continuous function w.r.t. the sup norm. Evaluated at convergent sequences $\{r_s, w_s\}$ with $r_s \to r^* < \beta^{-1} - 1$ and $w_s \to w^* > 0$,

(i) $\mathscr{X}(\{r_s, w_s\})$ converges to a limit $\mathscr{X}^{ss}(r^*, w^*)$ that only depends on the limits r^* and w^*.

(ii) $\mathscr{X}$ is Fréchet-differentiable in each of its two arguments, with β-quasi-Toeplitz derivatives $\mathbf{X}^{(r)}(\{r_s, w_s\}) : \ell^\infty \to \ell^\infty$ and $\mathbf{X}^{(w)}(\{r_s, w_s\}) : \ell^\infty \to \ell^\infty$ whose symbol vector only depends on the limits r^* and w^*.

Proof. Omitted. $\square$

The limit $\mathscr{X}^{ss}$ introduced in property (i) characterizes the steady state behavior of the variable $\mathscr{X}$ as a function of the steady state interest rate and the wage. We write $\mathscr{A}^{ss}(r, w)$, $\mathscr{N}^{ss}(r, w)$, $\mathscr{C}^{ss}(r, w)$, $\mathscr{U}^{ss}(r, w)$ for steady-state variables as a function of r and w.

1.3 Competitive Equilibrium and Implementability

Definition 1.4: Competitive Equilibrium

A competitive equilibrium in our economy is a collection of quantities $\{Y_t, N_t, B_t, C_t, A_t\}$, tax rates $\{\tau_t\}$, and prices $\{r_t, w_t\}$ such that:

1. The after-tax wage is given by (1.7).
2. Households optimize given prices: aggregate consumption C_t is given by (1.11), aggregate assets A_t are given by (1.9), and aggregate effective labor supply N_t is given by (1.10).
3. Output is given by (1.6).
4. The government budget constraint (1.8) holds.
5. The asset market clears, $A_t = B_t$, and the goods market clears, $C_t + G = Y_t$.

A competitive equilibrium is a steady state equilibrium if all quantities, tax rates, and prices are constant.

Proposition 1.5

$\{r_t, w_t\}$ are part of a competitive equilibrium iff

$$\mathscr{C}_t(\{r_s, w_s\}) + G = \mathscr{N}_t(\{r_s, w_s\}), \tag{1.15}$$

or alternatively, iff

$$G + (1 + r_t)\mathscr{A}_{t-1}(\{r_s, w_s\}) = \mathscr{A}_t(\{r_s, w_s\}) + (1 - w_t)\mathscr{N}_t(\{r_s, w_s\}). \tag{1.16}$$

Proof. **Necessity**

If $\{r_t, w_t\}$ are part of a competitive equilibrium, then clearly the goods market clearing condition (1.15) has to hold, once optimal consumption and labor supply have been substituted in. Likewise, the government budget constraint (1.16), with optimal asset demand $\mathscr{A}_t(\{r_s, w_s\})$ and labor supply $\mathscr{N}_t(\{r_s, w_s\})$, has to hold as well.

Sufficiency

Assume sequences $\{r_t, w_t\}$ satisfy equation (1.15), then we try to construct a competitive equilibrium with these sequences.

Define $\tau_t \equiv 1 - w_t$, $C_t \equiv \mathscr{C}_t(\{r_s, w_s\})$, $B_t \equiv A_t \equiv \mathscr{A}_t(\{r_s, w_s\})$, $N_t \equiv Y_t \equiv \mathscr{N}_t(\{r_s, w_s\})$. Then, these objects satisfy condition 1,2,3, and 5 of Definition 1.4.

Now we consider condition 4, the government budget constraint.

The optimal household policies must satisfy the consolidated household budget constraint,

$$\mathscr{C}_t(\{r_s, w_s\}) + \mathscr{A}_t(\{r_s, w_s\}) = (1 + r_t)\mathscr{A}_{t-1}(\{r_s, w_s\}) + w_t\mathscr{N}_t(\{r_s, w_s\}),$$

which is (1.2) integrated across households i.

Substituting out $\mathscr{C}_t$ using (1.15), we obtain equation (1.16), which is exactly condition 4.

This shows that (1.15) is sufficient for a competitive equilibrium. Reversing the procedure, (1.16) is also sufficient for a competitive equilibrium. $\square$

1.4 Infinitely Anticipated Shocks and Discounted Elasticities

Using the sequence-space function for asset demand (1.9), we see that the date $t = s + h$ response, in percentage points of steady state assets, is given by the derivative (semi-elasticity) $\partial \ln \mathscr{A}_{s+h}/\partial r_s$, where h runs from $-s$ to ∞. For s sufficiently large, these derivatives become independent of s, and only depend on the time horizon h relative to the date of the anticipated shock. Also, the derivatives (elasticities) of labor supply to an increase in labor income taxes (equivalent to a decrease in after-tax wages) can be written as $-\partial \ln \mathscr{N}_{s+h}/\partial \ln w_s$, with a large date s.

One way to summarize responses to infinitely anticipated shocks is to discount them relative to the period of the shock. For a general discount $\delta \in [\beta, 1]$[3], define the following discounted elasticities,

$$\epsilon^{A,r}(\delta) \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \ln \mathscr{A}_{s+h}}{\partial r_s},$$

and

$$\epsilon^{N,\tau}(\delta) \equiv -\epsilon^{N,w}(\delta) \equiv -\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \ln \mathscr{N}_{s+h}}{\partial \ln w_s},$$

and similarly $\epsilon^{U,w}(\delta)$, $\epsilon^{C,r}(\delta)$ and so on.

Now we explain that the discounted elasticities defined here are indeed well-defined in our model.

Lemma 1.6: Existence of Discounted Elasticities

Let $\mathscr{X}$ be a sequence-space function of sequences $\{r_s, w_s\}$. The discounted derivative

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{X}_{s+h}}{\partial r_s}$$

evaluated around constant paths $r_s = r$, $w_s = w$, is well-defined for $\delta \in [\beta, 1]$[a].

[a] The result for derivatives w.r.t. w_s follows analogously.

Proof. Because a constant path is obviously a convergent sequence, $\mathscr{X}$ is Fréchet-differentiable by Proposition 1.3. Represent it derivative by the β-quasi-Toeplitz matrix $\mathbf{M} = [M_{t,s}]$, which we decompose into

(i) The Toeplitz portion $\mathbf{T}(\mathbf{a}) \equiv [a_{t-s}]_{t,s=0}^{\infty}$, where $\mathbf{a}$ is the symbol vector as in Proposition 1.3.

(ii) The correction term $\mathbf{E} = [M_{t,s}]$, defined as $E_{t,s} = M_{t,s} - a_{t-s}$.

Then the derivative can be written as

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{X}_{s+h}}{\partial r_s} \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h M_{s+h,s} = \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h (a_h + E_{s+h,s}).$$

By Lemma 1.7, we have that

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h E_{s+h,s} = 0.$$

[3] The discount factor δ is left general for the moment. It will take the role of the social discount factor, so a natural baseline will be to assume it equals the private discount factor, $\delta = \beta$.

Thus,

$$\lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h M_{s+h,s} = \lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h a_h,$$

which exists by equation (1.13). This proves existence of the discounted elasticities. ☐

Lemma 1.7

In the proof of Lemma 1.6,

$$\lim_{s\to\infty}\sum_{h=-\infty}^{\infty}\delta^h |E_{s+h,s}| = 0,$$

where we use the notation $E_{j,s} = 0$ for $j < 0$[a].

[a] This sum is well-defined by (1.14), $\forall \delta \in [\beta, 1]$.

Proof. By equation (1.13), we have

$$\lim_{s\to\infty} M_{s+h,s} = a_h,$$

which implies that

$$\lim_{s\to\infty} |E_{s+h,s}| = \lim_{s\to\infty} |M_{s+h,s} - a_h| = 0.$$

By equation (1.14), $\exists D > 0, \psi \in (0,1)$, s.t.

$$|M_{t,s}| \le \begin{cases} D\psi^h & h \ge 0 \\ D(\beta\psi)^{-h} & h < 0 \end{cases}.$$

Similarly, for the symbol vector (as well as the limit of $M_{s+h,s}$), $\exists D' > 0, \psi \in (0,1)$, s.t.

$$|a_h| \le \begin{cases} D'\psi^h & h \ge 0 \\ D'(\beta\psi)^{-h} & h < 0 \end{cases}.$$

Thus,

$$|E_{s+h,s}| = |M_{s+h,s} - a_h| \le |M_{s+h,s}| + |a_h| \le \begin{cases} (D+D')\psi^h & h \ge 0 \\ (D+D')(\beta\psi)^{-h} & h < 0 \end{cases}.$$

Multiplying both sides by δ^h and taking the sum one yields

$$\begin{aligned}\sum_{h=-\infty}^{\infty}\delta^h |E_{s+h,s}| &\le (D+D')\left[\sum_{h=0}^{\infty}(\delta\psi)^h + \sum_{h=-\infty}^{-1}\delta^h(\beta\psi)^{-h}\right] \\ &= (D+D')\left[\sum_{h=0}^{\infty}(\delta\psi)^h + \sum_{h=1}^{\infty}\left(\frac{\beta}{\delta}\psi\right)^h\right].\end{aligned}$$

Since $\delta\psi < 1$ and $\beta\psi/\delta < 1$, this summable function converges. That is, $\delta^h|E_{s+h,s}|$ is dominated by a function which is independent of s.

Finally, by the Dominated Convergence Theorem, we have

$$\lim_{s\to\infty}\sum_{h=-\infty}^{\infty}\delta^h |E_{s+h,s}| = \sum_{h=-\infty}^{\infty}\delta^h \lim_{s\to\infty}|E_{s+h,s}| = 0.$$

☐

2 Sequence-Space Approach to Ramsey Taxation

2.1 Ramsey Problem

The full-commitment Ramsey planner maximizes utilitarian welfare each period, discounted at $\delta \in [\beta, 1)$,

$$\sum_{t=0}^{\infty} \delta^t \mathscr{U}_t(\{r_s, w_s\}) \tag{2.1}$$

by choice of the time path of prices $\{r_s, w_s\}$, s.t. either (1.15) or (1.16). It is simpler to work with the government budget constraint (1.16) as an implementability condition, and we do this from now on. The wage $w_s = 1 - \tau_s$ is controlled directly by the planner, and the interest rate r_s is endogenously determined by (1.16). Not bearing on any of our results, this problem allows the planner to choose the date-0 return r_0.

Let λ_t be the current-value multiplier on the implementability constraint. Define the Lagrangian:

$$\mathscr{L} = \sum_{h=-s}^{\infty} \delta^h \mathscr{U}_{s+h} + \sum_{h=-s}^{\infty} \delta^h \lambda^{s+h} [\mathscr{A}_{s+h} + (1 - w_{s+h})\mathscr{N}_{s+h} - G - (1 + r_{s+h})\mathscr{A}_{s+h-1}].$$

The FOC w.r.t. r_s:

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{U}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \lambda^{s+h} \left[\frac{\partial \mathscr{A}_{s+h}}{\partial r_s} + (1 - w_{s+h}) \frac{\partial \mathscr{N}_{s+h}}{\partial r_s} - (1 + r_{s+h}) \frac{\partial \mathscr{A}_{s+h-1}}{\partial r_s} \right] - \lambda_s \mathscr{A}_{s-1} = 0. \tag{2.2}$$

The FOC w.r.t. w_s:

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{U}_{s+h}}{\partial w_s} + \sum_{h=-s}^{\infty} \delta^h \lambda^{s+h} \left[\frac{\partial \mathscr{A}_{s+h}}{\partial w_s} + (1 - w_{s+h}) \frac{\partial \mathscr{N}_{s+h}}{\partial w_s} - (1 + r_{s+h}) \frac{\partial \mathscr{A}_{s+h-1}}{\partial w_s} \right] - \lambda_s \mathscr{N}_s = 0. \tag{2.3}$$

For any Ramsey plan $\{r_s, w_s\}$, these two necessary FOCs need to be satisfied.

2.2 Ramsey Steady State

Definition 2.1: Ramsey Steady State

A steady-state equilibrium consisting of quantities Y, N, B, C, A, a tax rate τ, and prices r, w is called a Ramsey steady state (RSS, hereafter) of the economy if there exists a Ramsey plan such that $\mathscr{C}_t \to C$, $\mathscr{A}_t \to A$, $\mathscr{N}_t \to N$, $w_t \to w$, $r_t \to r$.

The next lemma shows that we only need convergence of the two price sequences to define a RSS, convergence of all other equilibrium objects follows.

Lemma 2.2

A steady-state equilibrium with prices r, w is a RSS iff there is a Ramsey plan $\{r_s, w_s\}$ with $r_t \to r$ and $w_t \to w$.

Proof. **Necessity**

Satisfied directly by Definition 2.1.

Sufficiency

Take a steady state equilibrium with prices r, w. Suppose there is a Ramsey plan $\{r_s, w_s\}$ with $r_t \to r$ and $w_t \to w$. By Proposition 1.3, $\mathscr{C}_t \to \mathscr{C}^{ss}(r, w)$, $\mathscr{A}_t \to \mathscr{A}^{ss}(r, w)$, and $\mathscr{N}_t \to \mathscr{N}^{ss}(r, w)$. □

It is useful to define a unit-less measure of liquidity in any steady state of the economy,

$$\ell \equiv \frac{A}{wN},$$

as the ratio of available assets for self-insurance to aggregate after-tax income. Moreover, define the effective marginal rate of substitution between a one-time interest rate increase and a one-time wage increase around some steady state as

$$\mathrm{MRS}(\delta) \equiv \frac{\epsilon^{U.w}(\delta)}{\epsilon^{U.r}(\delta)}$$

Now we derive the discounted elasticities of utility when $\delta = \beta$, then the $\mathrm{MRS}(\beta)$.

Lemma 2.3

If $\delta = \beta$, then

$$\epsilon^{U,w}(\beta) = \lim_{s\to\infty} \sum_{h=-s}^{\infty} \beta^h \frac{\partial \mathcal{U}_{s+h}}{\partial \ln w_s} = w \int u_c(c_{i,t}, n_{i,t}) e_{i,t} n_{i,t} \mathrm{d}i, \tag{2.4}$$

and

$$\epsilon^{U,r}(\beta) = \lim_{s\to\infty} \sum_{h=-s}^{\infty} \beta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} = \int u_c(c_{i,t}, n_{i,t}) a_{i,t-1} \mathrm{d}i. \tag{2.5}$$

Proof. We focus on (2.4); (2.5) follows analogously.

Observe that by (1.1),

$$\sum_{h=-s}^{\infty} \beta^{s+h} \mathcal{U}_{s+h} = \int \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t}) \mathrm{d}i,$$

so that, around a steady state with r and w, taking the derivative w.r.t. $\ln w_s$ one yields

$$\sum_{h=-s}^{\infty} \beta^{s+h} \frac{\partial \mathcal{U}_{s+h}}{\partial \ln w_s} = \int \frac{\partial \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t})}{\partial \ln w_s} \mathrm{d}i = w \int \frac{\partial \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t})}{\partial w_s} \mathrm{d}i.$$

Applying the Envelope Theorem to (1.1) with budget constraint (1.2), we find that

$$\frac{\partial \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t})}{\partial w_s} = \beta^s \mathbb{E}_0 \frac{\partial u(c_{i,s}, n_{i,s})}{\partial w_s} = \beta^s \mathbb{E}_0 [u_c(c_{i,s}, n_{i,s}) e_{i,s} n_{i,s}].$$

Combining the two equations above, we have

$$\sum_{h=-s}^{\infty} \beta^{s+h} \frac{\partial \mathcal{U}_{s+h}}{\partial \ln w_s} = \beta^s w \int \mathbb{E}_0 [u_c(c_{i,s}, n_{i,s}) e_{i,s} n_{i,s}] \mathrm{d}i = \beta^s w \int u_c(c_{i,s}, n_{i,s}) e_{i,s} n_{i,s} \mathrm{d}i.$$

This directly implies (2.4). □

By Lemma 2.3, we obtain

$$\mathrm{MRS}(\beta) = \frac{w \int u_c(c_{i,t}, n_{i,t}) e_{i,t} n_{i,t} \mathrm{d}i}{\int u_c(c_{i,t}, n_{i,t}) a_{i,t-1} \mathrm{d}i} = \frac{1}{\ell} \cdot \frac{w \int u_c(c_{i,t}, n_{i,t}) \frac{e_{i,t} n_{i,t}}{N} \mathrm{d}i}{\int u_c(c_{i,t}, n_{i,t}) \frac{a_{i,t-1}}{A} \mathrm{d}i}. \tag{2.6}$$

2.3 RSS Condition: Converging Multiplier

Proposition 2.4

If a pair of prices (r, w) is part of a RSS of a Ramsey plan with converging Lagrange multipliers λ_t, then two conditions have to hold:

1. The steady state government constraint:

$$G + r\mathscr{A}^{ss}(r, w) = (1 - w)\mathscr{N}^{ss}(r, w). \tag{2.7}$$

2. Either the following RSS optimality condition,

$$\Big[(1-\delta(1+r))\cdot\mathscr{e}\cdot(\mathrm{MRS}\cdot\epsilon^{A,r}+\epsilon^{A,\tau})\Big]-\Big[\frac{1-w}{w}(-\epsilon^{N,r}-\mathrm{MRS}\cdot\epsilon^{N,r})\Big]-\Big[\mathscr{e}\cdot\mathrm{MRS}-1\Big]=0, \tag{2.8}$$

where we omitted δ as an argument of the elasticities and the MRS, or the unconstrained optimality conditions[a],

$$\epsilon^{U,r} = \epsilon^{U,w} = 0.$$

[a]We allow for this case for completeness, though mathematically this can never be satisfies if $\delta = \beta$, see the proof.

Proof. We begin with the trivial case. Let λ_t converge to some value λ. If $\lambda = 0$, the RSS must be at the unconstrained optimum, that is,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{U}_{s+h}}{\partial r_s} \to 0, \qquad \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{U}_{s+h}}{\partial w_s} \to 0.$$

To derive (2.8), we instead work with the case where $\lambda \neq 0$.

By Lemma 2.2, convergence of $\{r_t, w_t\}$ implies that all the curly functions converge to their steady state values, evaluated at (r, w). In particular, the implementability condition (1.16) must converge to the steady state government budget constraint as $t \to \infty$, which is exactly (2.7).

We begin deriving (2.8) by taking limits of the FOCs (2.2) and (2.3) as $s \to \infty$, and by simplifying the resulting expressions. Since these steps are almost exactly analogous for both FOCs, we focus on (2.2).

We first need to ensure that the limits of (2.2) exist. Since the first term

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathscr{U}_{s+h}}{\partial r_s} = \epsilon^{U,r}(\delta),$$

and the last term

$$\lim_{s\to\infty} \lambda_s \mathscr{A}_{s-1} = \lambda \mathscr{A}^{ss}(r, w),$$

we only need to consider the second term, or actually, arbitrary one sum of it.

Recall that matrix $\left[\frac{\partial \mathscr{A}_t}{\partial r_s}\right]_{t,s}$ is β-quasi-Toeplitz. Denote by $\mathbf{a} = (a_t)$ the symbol vector of this matrix and by $E_{t,s} \equiv \frac{\partial \mathscr{A}_t}{\partial r_s} - a_{t,s}$ its correction matrix. To introduce the multiplier, define $M_{t,s} \equiv \lambda_t \frac{\partial \mathscr{A}_t}{\partial r_s}$. It is trivial to show that $M_{t,s}$ is β-quasi Toeplitz. Thus, by Lemma 1.7, we have that

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \frac{\partial \mathscr{A}_{s+h}}{\partial r_s} = \sum_{h=-\infty}^{\infty} \delta^h \lambda a_h = \lambda \mathscr{A}^{ss}(r, w)\epsilon^{A,r}(\delta).$$

Analogous steps show that

$$\lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h\lambda_{s+h}(1-w_{s+h})\frac{\partial\mathscr{N}_{s+h}}{\partial r_s}=\lambda(1-w)\mathscr{N}^{ss}(r,w)\epsilon^{N,r}(\delta),$$

and

$$\lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h\lambda_{s+h}(1+r_{s+h})\frac{\partial\mathscr{A}_{s+h-1}}{\partial r_s}=\lambda\delta(1+r)\mathscr{A}^{ss}(r,w)\epsilon^{A,r}(\delta),$$

Taken together, the preceding limit analysis establishes that a limit of all the terms in (2.2) exists, with the limiting equation

$$\epsilon^{U,r}+\lambda\Big[A\epsilon^{A,r}+(1-w)N\epsilon^{N,r}-\delta(1+r)A\epsilon^{A,r}\Big]-\lambda A=0, \tag{2.9}$$

where we abbreviate assets and labor supply at the RSS as $A=\mathscr{A}^{ss}(r,w)$ and $N=\mathscr{N}^{ss}(r,w)$.

The limit of (2.3) can similarly found to be

$$\epsilon^{U,w}+\lambda\Big[A\epsilon^{A,w}+(1-w)N\epsilon^{N,r}-\delta(1+r)A\epsilon^{A,w}\Big]-\lambda Nw=0. \tag{2.10}$$

Using the definition for liquidity ℓ, we simplify (2.9) and (2.10), obtaining

$$\begin{aligned}\lambda^{-1}A^{-1}\ell\epsilon^{U,r}+(1-\delta(1+r))\ell\epsilon^{A,r}+\frac{1-w}{w}\epsilon^{N,r}-\ell=0,\\ \lambda^{-1}A^{-1}\ell\epsilon^{U,w}+(1-\delta(1+r))\ell\epsilon^{A,w}+\frac{1-w}{w}\epsilon^{N,w}-1=0.\end{aligned}$$

By eliminating λ from those two equations, we finally derive (2.8). □

We decompose the LHS of optimality condition (2.8) into three items, corresponding to different economic intuitions:

(i) $[(1-\delta(1+r))\cdot\ell\cdot(\mathrm{MRS}\cdot\epsilon^{A,r}+\epsilon^{A,\tau})]$, the **liquidity effect**. The planner can raise the present value of resources by issuing more debt when there is a gap between the social discount factor δ and the interest rate. Since $\beta(1+r)<1$ in any steady state, this term takes a positive sign whenever $\delta=\beta$, but it can be negative when the planner uses a relatively high δ.

(ii) $-\left[\frac{1-w}{w}(-\epsilon^{N,r}-\mathrm{MRS}\cdot\epsilon^{N,r})\right]$, the **labor supply**. It captures any disincentive effects on labor supply of greater labor income taxation and interest rates. One would intuit that this term is negative, indicating lower labor supply, but this will not always be the case.

(iii) $-(\ell\cdot\mathrm{MRS}-1)$, the **redistribution**. It demonstrates that combined increase in interest rates and labor income taxes has distributional consequences.

Proposition 2.5

Assume $u(c,n)$ is log-separable, as in (1.5). Then, condition (2.8) is equivalent to

$$(1-\delta(1+r))\ell\epsilon^{A,r}+\frac{1-w}{w}\epsilon^{N,r}+(r\ell+1)\epsilon^{U,r}-\ell=0. \tag{2.11}$$

Proof. Omitted. □

2.4 RSS Condition: Diverging Multiplier

Proposition 2.6

If a pair of prices (r, w) is part of a RSS of a Ramsey plan with diverging Lagrange multipliers, $\lambda_t \to \infty$ with $\lambda_t/\lambda_{t-1} \to \eta \in [1, \delta^{-1}]$, then the government budget constraint (2.7) as well as two optimality conditions

$$(1 - \delta\eta(1+r))\ell\epsilon^{A,\tau} - \frac{1-w}{w}(-\epsilon^{N,\tau}) + 1 = 0, \tag{2.12}$$
$$(1 - \delta\eta(1+r))\ell\epsilon^{A,r} - \frac{1-w}{w}(-\epsilon^{N,r}) - \ell = 0, \tag{2.13}$$

have to hold. Here, we omitted $\delta\eta$ as an argument of the elasticities.

Proof. Omitted. □

3 Quantitative Analysis

3.1 Solution Method and Calibration

With a converging multiplier, we determine the RSS as figure 1. Following the dual approach, we look for a pair of an interest rate r and a labor income tax rate $\tau = 1 - w$, on the y-axis and the x-axis, respectively, s.t.

(i) The government budget constraint (2.7) (black line) requires that higher r i.e., liquidity provision, which need to be financed by greater τ.

(ii) The optimality condition (2.8) (red line) requires that we optimally resolve the trade-off between r and τ.

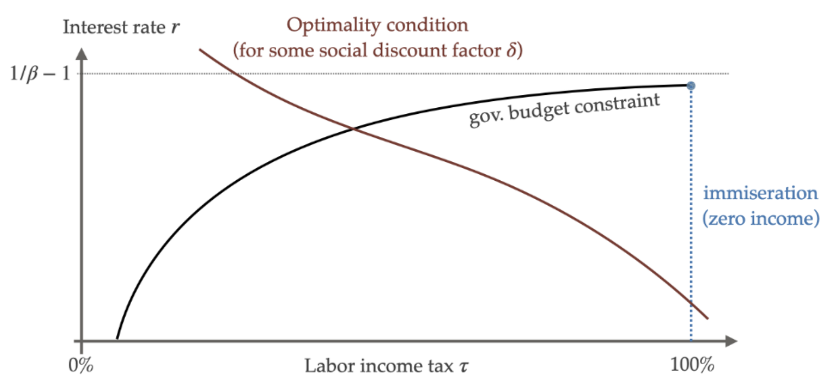

Figure 1: Determination of the RSS

Notice that the highest r consistent with the government budget constraint is strictly below the private discount rate $\beta^{-1} - 1$, because of positive government spending $G > 0$.

In this section, we focus on log-separable preferences (1.5) for all the quantitative exercises. And as a starting point, we use the following baseline calibration of the economy,

- The preferences $u(c, n) = \ln c - \ln n$, that is, $v(n)$ is isoelastic with a Frisch elasticity of 1.

- The idiosyncratic productivities $e_{i,t} = 0.90 e_{i,t-1} + \varepsilon_t$, with $\sigma_\varepsilon = 0.92$.
- The initial steady state of the economy is $B/Y = 100\%$, $G/Y = 20\%$. This implies $\beta = 0.897$ to satisfy asset market clearing, and $\tau = 22\%$ to satisfy the government budget constraint.

3.2 Searching for a RSS

Initially, we set $\delta = \beta$ as a baseline. We search for a RSS by varying τ, then solving for the associated r that satisfies (2.7), and finally by evaluating (2.8) for that pair of τ and r. Since $w = 1 - \tau$, we can think of the approach as walking from the bottom left to the top right corner along the black line in figure 1, always evaluating the optimality condition (2.8), seeing whether it ever crosses zero.

Figure 2 plots the three terms on the LHS of (2.8) as well as the sum of all terms. As expected, the liquidity effect is positive, and the redistribution is negative. But the labor supply effect is positive, indicating that $\tau \uparrow$ i.e., $r \uparrow$ lead to $\delta^t n_t \uparrow$. Moreover, this prevents the sum of all three effects from ever crossing zero. Thus, there is no RSS with a converging multiplier in this economy. We have checked the FOCs with exploding multipliers and do not find a RSS, too.

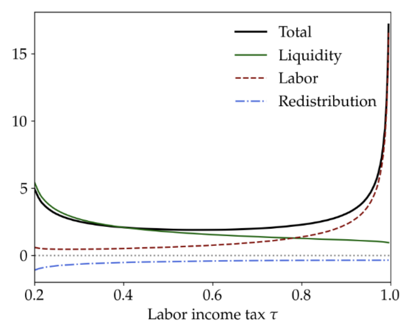

Figure 2: Optimality condition with $\delta = \beta$

We next set $\delta = 1$. This can be interpreted as a planner that is maximizing steady state welfare only, ignoring the transition it takes to reach the RSS.

Figure 3 plots the calculation. Unlike the baseline result, liquidity benefit is now negative, and the redistribution is now positive. The labor supply now go in the intuitive direction and turn negative eventually. This ensures a unique intersection of (2.8) with zero, and hence a unique optimal steady state (OSS, hereafter) in this economy.

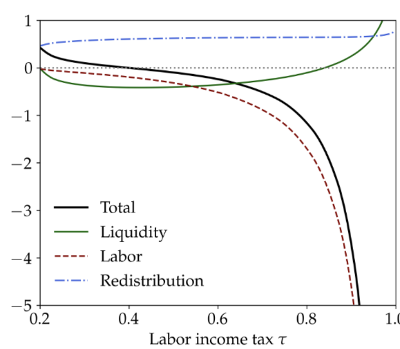

Figure 3: Optimality condition with $\delta = 1$

Figure 4 follows this approach, and computes various outcomes, whenever a candidate RSS for a given $\delta \in [\beta, 1)$ was found[4]. As δ is moved from 1 to β, many quantities and prices approach a corner. The grey-dotted vertical line shows the result with $\delta = \beta$. In this case, the household tends to immiseration: the labor tax rates $\tau_t \to 100\%$, and after-tax wages $w_t \to 0$.

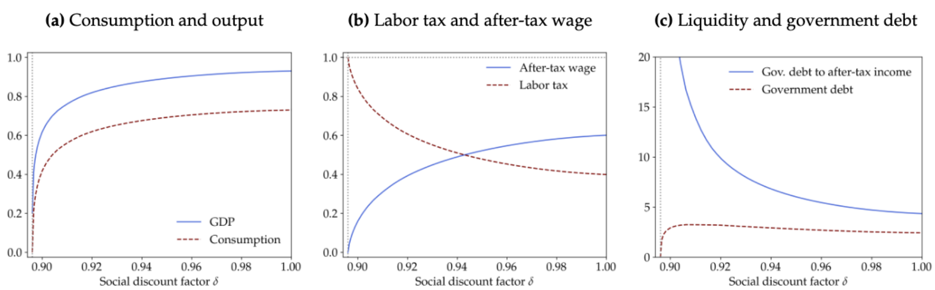

Figure 4: RSS as function of $\delta \in [\beta, 1)$

4 Immiseration Results with Different Preferences

4.1 Baseline: Log-Separable Preferences

Motivated by figure 4, we next provide a necessary condition that specifically captures immiseration dynamics in the case of log-separable preferences (1.5). To prove our main result, we firstly introduce the detrending approach worked for the sequence space functions, with KPR preferences.

Lemma 4.1: Detrending

Assume $u(c, n)$ is balanced growth compatible, as in (1.4). $\forall \{r_t, w_t\}$, $\forall \gamma \in (0, 1)$,

$$\mathscr{U}_t(\{r_s, w_s\}, \beta) = \gamma^{(1-\sigma)t} \mathscr{U}_t\left(\left\{\frac{1+r_s}{\gamma} - 1, w_s\gamma^{-s}\right\}, \beta\gamma^{1-\sigma}\right),$$
$$\mathscr{A}_t(\{r_s, w_s\}, \beta) = \gamma^t \mathscr{A}_t\left(\left\{\frac{1+r_s}{\gamma} - 1, w_s\gamma^{-s}\right\}, \beta\gamma^{1-\sigma}\right),$$
$$\mathscr{N}_t(\{r_s, w_s\}, \beta) = \mathscr{N}_t\left(\left\{\frac{1+r_s}{\gamma} - 1, w_s\gamma^{-s}\right\}, \beta\gamma^{1-\sigma}\right).$$

Proof. Rewriting the household problem that is aggregated by sequence-space functions $\mathscr{U}$, $\mathscr{A}$, $\mathscr{N}$,

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t})\right],$$

s.t.

$$c_{i,t} + a_{i,t} = (1 + r_t)a_{i,t-1} + w_t e_{i,t} n_{i,t},$$
$$a_{i,t} \geq 0.$$

[4]In fact, there is only a small interval of positive measure $[\beta, \tilde{\beta})$ just above β in which δ do not give rise to a candidate RSS.

Define $\hat{c}_{it} \equiv c_{it}/\gamma^t$ and $\hat{a}_{it} \equiv a_{it}/\gamma^t$ as detrended consumption and asset holdings. By dividing γ^t, the problem can be represented as

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \gamma^{(1-\sigma)t} u(\hat{c}_{i,t}, n_{i,t}) \right],$$

s.t.

$$\hat{c}_{i,t} + \hat{a}_{i,t} = \frac{1+r_t}{\gamma} \hat{a}_{i,t-1} + \gamma^{-t} w_t e_{i,t} n_{i,t},$$
$$\hat{a}_{i,t} \geq 0.$$

This immediately implies the relationships in the lemma. □

Proposition 4.2

Assume $u(c, n)$ is of the log-separable form, (1.5). Let $\{r_t, w_t\}$ be an optimal Ramsey plan s.t.

(i) $w_t \to 0$ with asymptotic decay factor $\gamma \in [\delta, 1)$, that is, $\exists \lim w_t/\gamma^t > 0$.

(ii) $r_t \to r < \gamma/\beta - 1$.

(iii) $\lambda_t \to \pm\infty$ with asymptotic factor $\eta \in (1, \delta^{-1})$, that is, $\exists \lim \lambda_t/\eta^t > 0$.

Then, the following two conditions have to hold, evaluated at a detrended steady state with interest rate $1 + \hat{r} = (1 + r)/\gamma$, the same discount factor β, and wage $\hat{w} = 1$:

1. An immiseration-adjusted government budget constraint

$$G = \mathscr{N}^{ss}(\hat{r}, \hat{w}). \tag{4.1}$$

2. An immiseration-adjusted optimality condition

$$\epsilon^{N,\tau}(\delta\eta) = \epsilon^{N,r}(\delta\eta) = 0. \tag{4.2}$$

Proof. Consider the FOCs (2.2) and (2.3). For simplicity, again, we focus on (2.2). We detrend this equation using Lemma 4.1 by defining $1 + \hat{r}_s \equiv (1 + r_s)/\gamma$, $1 + \hat{r} \equiv (1 + r)/\gamma$, $1 + \hat{w}_s \equiv w_s/\gamma^s$, and all sequences-space functions with a hat, $\hat{\mathscr{X}}$ if they are evaluated around the detreded sequences $\{r_s, w_s\}$.

Then, (2.2) becomes

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \hat{\mathscr{U}}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \lambda^{s+h} \left[\gamma^{s+h} \frac{\partial \hat{\mathscr{A}}_{s+h}}{\partial r_s} + (1 - \hat{w}_{s+h}\gamma^{s+h}) \frac{\partial \hat{\mathscr{N}}_{s+h}}{\partial r_s} - \gamma^{s+h-1}(1 + r_{s+h}) \frac{\partial \hat{\mathscr{A}}_{s+h-1}}{\partial r_s} \right] - \lambda_s \gamma^{s-1} \hat{\mathscr{A}}_{s-1} =$$

Dividing by λ_s one yields

$$\frac{1}{\lambda_s} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \hat{\mathscr{U}}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \frac{\lambda^{s+h}}{\lambda_s} \left[\gamma^{s+h} \frac{\partial \hat{\mathscr{A}}_{s+h}}{\partial r_s} + (1 - \hat{w}_{s+h}\gamma^{s+h}) \frac{\partial \hat{\mathscr{N}}_{s+h}}{\partial r_s} - \gamma^{s+h}(1 + \hat{r}_{s+h}) \frac{\partial \hat{\mathscr{A}}_{s+h-1}}{\partial r_s} \right] - \gamma^{s-1} \hat{\mathscr{A}}_{s-1} = 0$$

As $s \to \infty$, by assumptions, all terms converges to zero except for the $\frac{\partial \hat{\mathscr{N}}_{s+h}}{\partial r_s}$ term, since $w_t \to 0$, which approaches

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} \frac{\partial \hat{\mathscr{N}}_{s+h}}{\partial r_s} = \sum_{h=-\infty}^{\infty} (\delta\eta)^h \frac{\partial \hat{\mathscr{N}}_{s+h}}{\partial r_s} = \epsilon^{N,r}(\delta\eta) = 0,$$

when evaluated around a detrended steady state with $\hat{w} = 1$ and $1 + \hat{r} = (1 + r)/\gamma$.

Using the same logic, (2.3) implies that $\epsilon^{N,\tau}(\delta\eta) = 0$. Then (4.2) is proved.

Finally, (2.7) is given by

$$G + \gamma(1 + \hat{r}_t)\gamma^{t-1}\mathscr{A}_{t-1}(\{\hat{r}_s, \hat{w}_s\}) = \gamma^t \mathscr{A}_t(\{\hat{r}_s, \hat{w}_s\}) + (1 - \gamma^t \hat{w}_t)\mathscr{N}_t(\{\hat{r}_s, \hat{w}_s\}).$$

Taking the limit of $t \to \infty$, we can find equation (4.1). □

In this detrended economy, the government budget constraint (2.7) collapses to (4.1). Given any δ, the optimal condition (4.2) can then be used to solve for η. A limitation of this proposition is that is does not allow us to back out the rate at which the economy tends to immiseration. Instead, (4.1) and (4.2) are two conditions that determine the detrended interest rate $\hat{r}$ as well as the factor η.

4.2 Additively Separable Preferences

We now start considering classes of preferences other than the log-separable ones used in the previous section.

Beginning by considering additively separable preferences,

$$u(c, n) = \frac{c^{1-\sigma} - 1}{1 - \sigma} - \phi \frac{n^{1+\nu}}{1 + \nu},$$

where $\phi > 0$ is a constant and ν is the inverse of the Frisch elasticity of labor supply. For $\sigma \neq 1$, these preferences are no longer compatible with balanced growth. We consider two different cases distinguished by the value of EIS, as in figure 5.

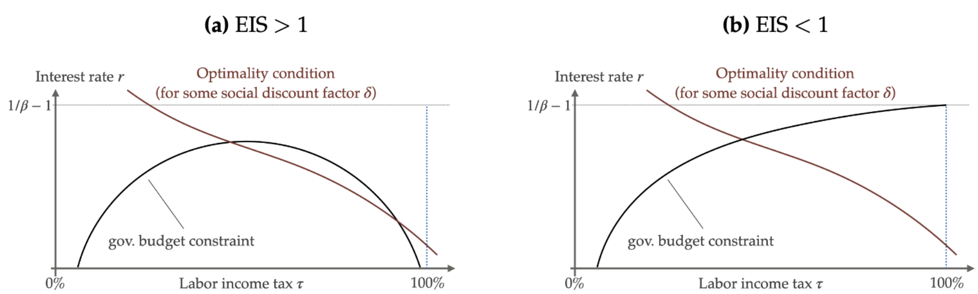

Figure 5: RSS conditions with additively separable preferences

In panel (a) with an EIS>1, the static substitution effect from wage changes is stronger than the static income effect. Thus, $\tau \uparrow$, $w \downarrow$, $n \downarrow$. As the labor tax gets closer to 100%, $\tau n \downarrow$, $r \downarrow$. The government budget constraint exhibits a hump-shaped profile.

With low government spending G, i.e., higher budget constraint line, there are in principle two candidate RSS, $\forall \delta \geq \beta$. When G becomes higher, there may be no any intersection between the two curves.

In panel (b) with an EIS<1, we have the opposite: $\tau \uparrow$, $n \uparrow$, $\tau n \uparrow$, $r \uparrow$. There always exists a unique candidate RSS, albeit one with labor income taxes incredibly close to 100%, suggesting the economy is going to near-immiseration in this case. See figure 6 for example, where we choose $\sigma = 2$.

4.3 GHH Preferences

We next move on to study Greenwood et al. (1988) preferences, or GHH preferences for short:

$$u(c, n) = \frac{\left(c - \phi \frac{n^{1+\nu}}{1+\nu}\right)^{1-\sigma} - 1}{1 - \sigma}.$$

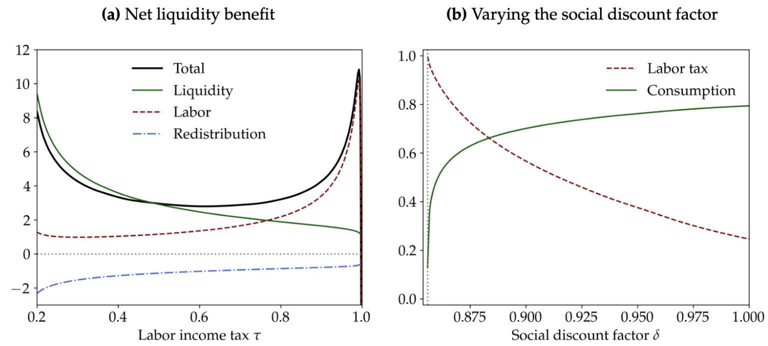

Figure 6: RSS for additively separable preferences with EIS<1

These preferences are designed to feature no income effects on labor supply, so that they are conceptually similar to additively separable preferences with an EIS>1.

With $G/Y = 0.25$, we find two potential RSS (figure 7a). With slightly higher value, $G/Y = 0.31$, there is no RSS with a converging multiplier (figure 7a).

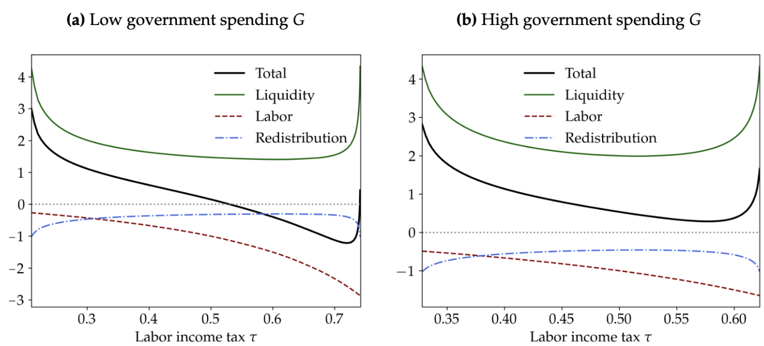

Figure 7: RSS for GHH preferences

4.4 KPR Preferences

Lastly, we consider the balanced-growth compatible (KPR) preferences, as in (1.4). Figure 8 considers the case of $\sigma = 2$. We do find a RSS, but the associated tax rate is very high, above 90%, towards near-immiseration.

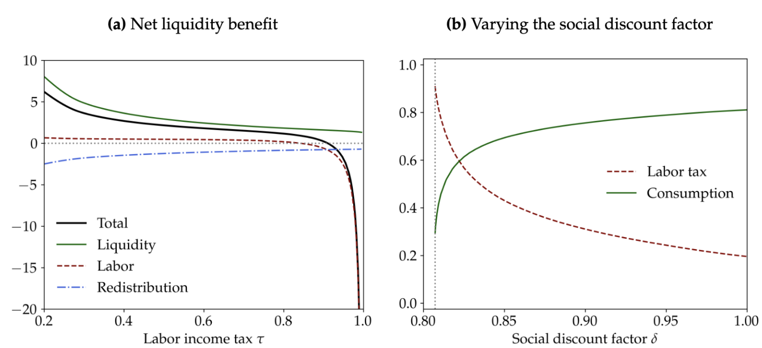

Figure 8: RSS for KPR preferences with EIS<1