Advanced Macroeconomics I
Lecture Notes

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Chapter 1

Introduction

1.1 What We Concern

Macroeconomics as a field is a child of the Great Depression in the 1930s, with the publication of John Maynard Keynes' *General Theory of Employment, Interest, and Money*. John Hicks offered a graphical interpretation of Keynes' work, that is IS-LM model, and it quickly became a workhorse model for macroeconomic policy (Hicks, 1937).

But it is universally known that, individual condition cannot explain the whole social condition, vice versa. For instance, entrepreneur management diametrically opposed to government (political) decision. Unfortunately, There was no such microeconomic behavior at the foundation of the first-generation Keynesian models; instead, decision rules for investment, consumption, labor supply, etc. were assumed rather than derived. This was not the solution to a household's optimization problem, but rather just seemed "natural". Paul Friedman's limits of monetary policy was a valid critique of the crudest versions of Keynesianism, but it was only the beginning of what was to come.

Robert Lucas launched a devastating critique on using these econometric models based on Keynesianism for policy evaluation (Lucas, 1976). He showed that while the *ad hoc* models might fit the data well, one cannot validly analyze the effects of policy within them.

Lucas and his followers contended that individuals maximizing their utility or firms maximizing their profits would also optimize their expectations. What does it mean to "optimize" expectations? In Lucas' framework it means that households use all available information to them when making their forecasts. This has come to be known as "rational expectations" and is now ubiquitous in modern macroeconomics and neoclassical growth theory. The evolution of economy is determined in a general equilibrium model with households maximizing lifetime utility and firms maximizing profits, that is how we analyze our problem.

The long-term problem, which can be said as economic growth, is undoubtedly the most fascinating part of the field of macroeconomics. The Solow growth model, developed by Solow (1956) and Swan (1956), was the first modern growth model (though it is not microfounded) and was built upon the Keynesian Harrod-Domar model. But in this model, the supply of capital stock is essentially fixed because the savings rate is exogenous and constant. Relaxing this assumption, Ramsey (1928), Cass (1965) and Koopman (1965) developed the RCK growth model, which is the benchmark model we concern in this lecture note.

Hayek (1945) argued the importance of the knowledge of the particular circumstances of time and place, and distinguished the main differences between planned economy and market economy. So in neoclassical growth model, we start from the case of decentralized markets, then comparing to the social planner's problem, and analyze the welfare.

When we successfully construct the model, there is always an unavoidable question: will economic growth continue indefinitely? So that we turned to analyze the stability of this model: the existence of steady state (will it convergence? and where?), the local stability on the steady state point, and then extending to global stability (how does it convergence?). Lastly we will show how to use computer programming to illustrate the accurate growth path of the model.

In "The Macroeconomist as Scientist and Engineer", Gregory Mankiw said: "I am reminded that the subfield of macroeconomics was born not as a science but more as a type of engineering. God put macroeconomists on earth not to propose and test elegant theories but to solve practical problems (Mankiw, 2006)." The purpose of this course is: pragmatism, pragmatism, and pragmatism. And what we really concern are the major question and theoretical tools on economic growth and development.

1.2 Typology

There are some characterizations that the neoclassical growth model (what we will discuss in this lecture note) has: homogeneous agents, infinite lifetime, continuous time, sequential view and deterministic. Now we gives a rough typology about macroeconomic models.

- Representativity.
  - Representative agents, such as the consumers who have identical preferences.
  - Heterogeneous agents, more complicated but more real.
- Time horizon.
  - Infinite lifetime, such as neoclassical growth model.
  - Finite lifetime, such as overlapping generations model.
- Measurement of time.
  - Discrete time, which is convenient to use computer programming.
  - Continuous time, which is convenient to use calculus techniques.
- Dynamic views.
  - Time-0 (sequential) view, the state of period 0 determines the optimal state of all periods (using discounted factors).
  - Recursive view, the status of the current period determines the status of the next period.
- Uncertainty.
  - Deterministic.
  - Stochastic.

Chapter 2

Mathematical Preliminaries

2.1 Lagrangian Method

Let us begin with an unconstrained optimization problem [1]:

$$\max_x f(x)$$

The first-order condition is

$$\nabla f = 0$$

Secondly, we consider the following optimization problem with equality constraints:

$$\begin{aligned} &\max_x f(x) \\ \text{s.t.} \quad & g(x) = 0 \end{aligned}$$

The first-order condition is

$$\nabla f + \lambda \nabla g = \nabla(f + \lambda g) = 0$$

where $\lambda > 0$ is the Lagrange multiplier.

Lastly, we consider the following optimization problem with inequality constraints:

$$\begin{aligned} &\max_x f(x) \\ \text{s.t.} \quad & g(x) \geq 0 \end{aligned}$$

We also have the first-order condition $\nabla(f + \lambda g) = 0$, and at the same time, we need another condition to discuss the property of the inequality, which is called the complementary slackness condition:

$$\lambda g(x) = 0$$

[1] In this chapter, we always assume $x \in \mathbb{R}^n$.

Theorem 2.1.1: Lagrangian Theorem

Let $h : \mathbb{R}^m \to \mathbb{R}, f : \mathbb{R}^m \to \mathbb{R}^n$ be C^1 functions, $C \in \mathbb{R}^n$, and $M = \{f = C\} \subset \mathbb{R}^m$. Assume that $\forall x \in M$, $\text{rank}(Df_x) = n$. If h attains a constrained local extremum at a, s.t. $f = C$, then $\exists \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$, such that

$$\nabla h(a) = \sum_{i=1}^{n} \lambda_i \nabla f_i(a)$$

Then we turn to a dynamic maximization problem using the Lagrangian method. Given the wage price w_t and interest price r_t, the household solves

$$\begin{aligned} &\max_{c_t, a_{t+1}} \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + a_{t+1} = w_t + (1 + r_t)a_t \end{aligned}$$

where c is the consumption and a is the asset, with $c_t \geq 0$, $a_{t+1} \geq 0$, $a_0 > 0$ given. For simplicity, we assume $u(c_t)$ satisfies the Inada condition. Then $\forall t$, $c_t > 0$. $T \to \infty$ is allowed.

The Lagrangian can be written as

$$\mathscr{L} = \sum_{t=0}^{T} \{\beta^t u(c_t) + \lambda_t \cdot [w_t + (1 + r_t)a_t - c_t - a_{t+1}] + \gamma_t a_{t+1}\}$$

where λ_t and γ_t are the Lagrange multipliers.

Lagrangian method is typically used for static models. In order to transform this dynamic problem into a static one, we extract the intertemporal variable a_{t+1}:

$$\begin{aligned} \mathscr{L} = ... &+ \beta^t u(c_t) + \lambda_t \cdot [w_t + (1 + r_t)a_t - c_t - a_{t+1}] + \gamma_t a_{t+1} \\ &+ \beta^{t+1} u(c_{t+1}) + \lambda_{t+1} \cdot [w_{t+1} + (1 + r_{t+1})a_{t+1} - c_{t+1} - a_{t+2}] + \gamma_{t+1} a_{t+2} \\ &+ ... \end{aligned}$$

So the first-order conditions are

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial c_t} &= \beta^t \frac{\partial u}{\partial c_t} - \lambda_t = 0, \quad t = 0, 1, ..., T \\ \frac{\partial \mathscr{L}}{\partial a_{t+1}} &= \begin{cases} -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) +] + \gamma_t = 0, & t = 0, 1, ...T - 1 \\ -\lambda_t + \gamma_t = 0 & t = T \end{cases} \end{aligned}$$

with the complementary slackness condition:

$$\gamma_t a_{t+1} = 0, \quad t = 0, 1, ...T$$

When $t = T$, we have $\gamma_T a_{T+1} = 0$. Since $\lambda_T = \gamma_T$, $\lambda_T = \beta^T \frac{\partial u}{\partial c_T} = \beta^T u'(c_T)$, we know that

$$\beta^T u'(c_T) a_{T+1} = 0$$

When $T \to \infty$, we have

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0$$

We call the two equations above the transversality condition.

2.2 Optimal Control

We continue to consider the same dynamic choice problem as in the last section, rewrite the Lagrangian (assume that $T \to \infty$, and a_{t+1} is strictly positive), and then rearrange this equation:

$$\begin{aligned}
\mathscr{L} &= \sum_{t=0}^{\infty} \{\beta^t u(c_t) + \lambda_{t+1}[w_t + (1+r_t)a_t - c_t - a_{t+1}]\} \\
&= u(c_0) + \lambda_1[w_0 + (1+r_0)a_0 - c_0 - a_1] \\
&+ \beta u(c_1) + \lambda_2[w_1 + (1+r_1)a_1 - c_1 - a_2] + ... \\
&= u(c_0) + \lambda_1[w_0 + r_0 a_0 - c_0] + \lambda_1 a_0 - \lambda_1 a_1 \\
&+ \beta u(c_1) + \lambda_2[w_1 + r_1 a_1 - c_1] + \lambda_2 a_1 - \lambda_2 a_2 + ... \\
&= u(c_0) + \lambda_1[w_0 + r_0 a_0 - c_0] + \lambda_1 a_0 \\
&+ \sum_{t=1}^{\infty} \{\beta^t u(c_t) + \lambda_{t+1} \cdot [w_t + (1+r_t)a_t - c_t - a_{t+1}]\} \\
&- \lambda_1 a_1 + \lambda_2 a_1 - \lambda_2 a_2 + ... \\
&= u(c_0) + \lambda_1[w_0 + r_0 a_0 - c_0] + \lambda_1 a_0 \\
&+ \sum_{t=1}^{\infty} \{\beta^t u(c_t) + \lambda_{t+1} \cdot [w_t + (1+r_t)a_t - c_t - a_{t+1}]\} + \sum_{t=1}^{\infty} a_t(\lambda_{t+1} - \lambda_t) \\
&= u(c_0) + \lambda_1[w_0 + r_0 a_0 - c_0] + \lambda_1 a_0 \\
&+ \sum_{t=1}^{\infty} \{\beta^t u(c_t) + \lambda_{t+1}[w_t + (1+r_t)a_t - c_t - a_{t+1}] + a_t(\lambda_{t+1} - \lambda_t)\}
\end{aligned}$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial c_t} &= \frac{\partial}{\partial c_t}[\beta^t u(c_t) + \lambda_{t+1}(r_t a_t + w_t - c_t)] = \beta^t u'(c_t) - \lambda_{t+1} = 0, \quad t = 0, 1, ... \\
\frac{\partial \mathscr{L}}{\partial a_t} &= \frac{\partial}{\partial a_t}[\beta^t u(c_t) + \lambda_{t+1}(r_t a_t + w_t - c_t)] + (\lambda_{t+1} - \lambda_t) = 0, \quad t = 0, 1, ...
\end{aligned}$$

Define a new function $\mathscr{H}_p$, called the present value Hamiltonian, by

$$\mathscr{H}_p = \beta^t u(c_t) + \lambda_{t+1}(r_t a_t + w_t - c_t)$$

Then the constraint can be simply written as

$$\frac{\partial \mathscr{H}_p}{\partial \lambda_{t+1}} = r_t a_t + w_t - c_t = a_{t+1} - a_t$$

and the FOCs can be written as

$$\frac{\partial \mathscr{H}_p}{\partial c_t} = 0$$
$$-\frac{\partial \mathscr{H}_p}{\partial a_t} = \lambda_{t+1} - \lambda_t$$

In continuous time, define $\beta = \frac{1}{1+\rho}$ as the one-period discount factor. Hence, the present value Hamiltonian becomes

$$\mathscr{H}_p = \mathrm{e}^{-\rho t} u(c) + \lambda(ra + w - c)$$

where λ is the present value co-state variable. And the FOCs can be written as

$$\frac{\partial \mathscr{H}_p}{\partial c} = 0$$
$$\frac{\partial \mathscr{H}_p}{\partial a} = -\dot{\lambda}$$

Eliminating the discount factor, the current value Hamiltonian is defined as

$$\mathscr{H}_c = u(c) + \mu(ra + w - c)$$

where μ is the current value co-state variable. Essentially it is $\mathscr{H}_p = \mathrm{e}^{-\rho t}\mathscr{H}_c$, with $\lambda = \mathrm{e}^{-\rho t}\mu$.

The FOCs become

$$\frac{\partial \mathscr{H}_p}{\partial c} = \frac{\partial \mathrm{e}^{-\rho t}\mathscr{H}_c}{\partial c} = \mathrm{e}^{-\rho t}\frac{\partial \mathscr{H}_c}{\partial c} = 0$$
$$\frac{\partial \mathscr{H}_p}{\partial a} = \mathrm{e}^{-\rho t}\frac{\partial \mathscr{H}_c}{\partial a} = -\dot{\lambda} = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\rho t}\mu = \mathrm{e}^{-\rho t}(\dot{\mu} - \rho\mu)$$

Rearranging the two equations above, we obtain

$$\frac{\partial \mathscr{H}_c}{\partial c} = 0$$
$$\frac{\partial \mathscr{H}_c}{\partial a} = \rho\mu - \dot{\mu}$$

2.3 Two-Dimensions Linear System

Consider a linear homogeneous system $\dot{x} = Ax$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2. The general solution is thus $c_1\mathrm{e}^{\lambda_1 t}v_1 + c_2\mathrm{e}^{\lambda_2 t}v_2$, where $c_1, c_2 \in \mathbb{R}$.

The eigenfunction is defined as

$$0 = |A - \lambda I| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$
$$= (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc$$

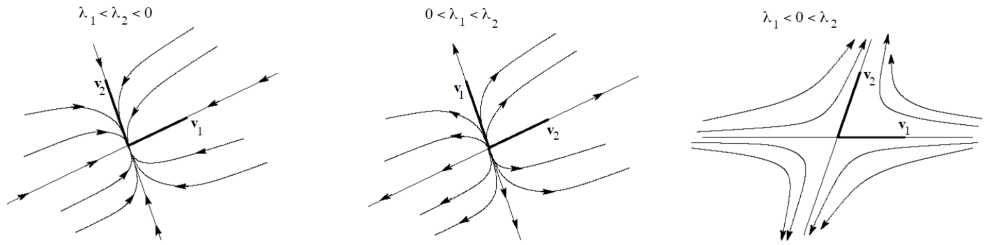
According to Vieta's formulas, we have

$$\lambda_1 + \lambda_2 = a + d = \tau(A)$$
$$\lambda_1 \lambda_2 = ad - bc = \Delta(A)$$

We classify the states of the linear system into the following categories, and assume that the two eigenvalues $\lambda_1 \leq \lambda_2$ (when $\lambda_1, \lambda_2 \in \mathbb{R}$), and the discriminant of this quadratic equation $\mathscr{D} = \tau^2 - 4\Delta$:

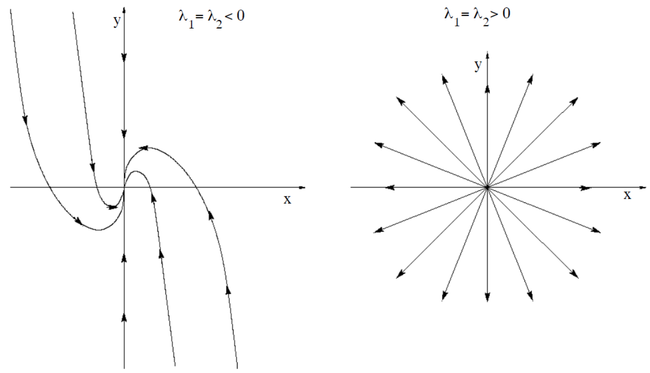
1. If $\Delta < 0$, we have two real eigenvalues $\lambda_1 < 0 < \lambda_2$. Then the linear system is a saddle.
2. If $\Delta > 0$ and $\tau > 0$, then the linear system is unstable.
   (a) If $\mathscr{D} > 0$, we have two real eigenvalues $0 < \lambda_1 < \lambda_2$. We call it an unstable node.
   (b) If $\mathscr{D} = 0$, we have two real eigenvalues $0 < \lambda_1 = \lambda_2$. We call it a degenerate unstable node.
   (c) If $\mathscr{D} < 0$, we have two complex eigenvalues λ_1, λ_2, and $\mathrm{Re}(\lambda_i) > 0$, $i = 1, 2$. We call it an unstable focus.
3. If $\Delta > 0$ and $\tau < 0$, then the linear system is (asymptotically) stable.
   (a) If $\mathscr{D} > 0$, we have two real eigenvalues $\lambda_1 < \lambda_2 < 0$. We call it a stable node.
   (b) If $\mathscr{D} = 0$, we have two real eigenvalues $\lambda_1 = \lambda_2 < 0$. We call it a degenerate stable node.
   (c) If $\mathscr{D} < 0$, we have two complex eigenvalues λ_1, λ_2, and $\mathrm{Re}(\lambda_i) < 0$, $i = 1, 2$. We call it a stable focus.
4. If $\Delta > 0$ and $\tau < 0$, then at least one of the eigenvalues is zero.

The phase diagrams of those states above can be demonstrated as



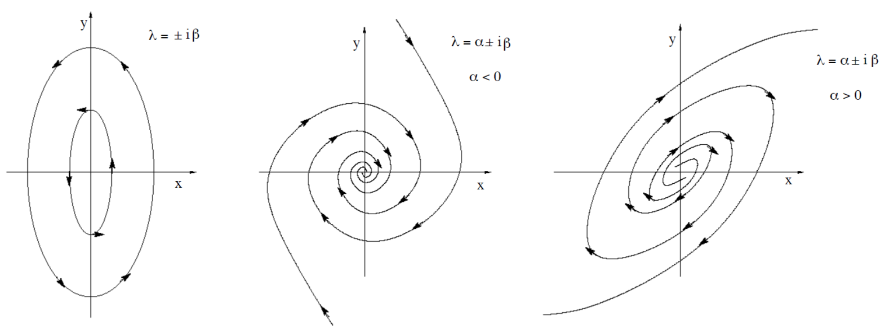
Left: a stable node. Middle: an unstable node. Right: a saddle.

Figure 2.1: Real Eigenvalues $\lambda_1 \neq \lambda_2$



Left: a stable degenerate node (in the case of only one linearly independent eigenvector).
Right: an unstable degenerate node (in the case of two linearly independent eigenvectors).

Figure 2.2: Real Eigenvalues $\lambda_1 = \lambda_2$



Left: a center. Middle: a stable spiral point. Right: an unstable spiral point.

Figure 2.3: Complex Eigenvalues λ_1, λ_2

2.4 Root Approximation of Nonlinear Equation

Theorem 2.4.1: Bolzano-Cauchy First Theorem

For a continuous function $f(x) \in [a, b]$, if $f(a) \cdot f(b) < 0$, then $\exists x_0 \in [a, b] : f(x_0) = 0$.

One of the first numerical methods developed to find the root of a nonlinear equation $f(x) = 0$ was the bisection method (also called the binary-search method).

Suppose that f is a continuous function on the interval $[a, b]$ and $f(a)f(b) < 0$. By Bolzano-Cauchy first theorem, f has at least one zero in the interval $[a, b]$.

We next calculate $m = (a + b)/2$ and test $f(m)$. If $f(m) = 0$, then m is the root and we are done. If not, then either $f(a)f(m) < 0$ or $f(b)f(m) < 0$.

In the former case, a root lies in $[a, m]$ and we rename m as b and do the same process. In the latter case, we rename m as a and continue the same process. The root now lies in a interval whose length is half of the length of the original interval. The process is repeated and we stop the iteration when $f(m)$ is very nearly zero or length of the interval $[a, b]$ is very small.

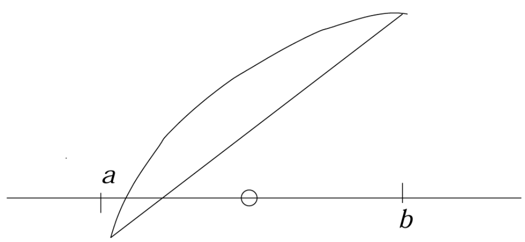


Figure 2.4: Illustration of Bisection Method

Consider the figure in which the root lies between a and b. In the first iteration of bisection method, the approximation lies at the small circle. However, since $|f(a)|$ is small, we expect the root to lie near a. This can be achieved if we joint the coordinates $(a, f(a))$ and $(b, f(b))$ and take the intersection c of the line with the x axis as the first approximation. Hence the new approximation c satisfies

$$\frac{c-a}{b-a} = \frac{0 - f(a)}{f(b) - f(a)}$$

Then we have

$$c = \frac{af(b) - bf(a)}{b - a}$$

If $f(c) = 0$, then c is the exact root. Otherwise we take $b = c$ if $f(a)f(c) < 0$ and $a = c$ if $f(c)f(b) < 0$. This process is then repeated. We call it regula falsi method.

Chapter 3

General Equilibrium Framework

At the beginning of this chapter, We will review some of the most important theoritical findings in 20th century economics.

Arrow and Debreu (1954) proofs the existence of an equilibrium are given for a competitive economy, which is the basis of general equilibrium framework, also the modern macroeconomics. We shows a simple version of this model.

Definition 3.0.1: Equilibrium Price Vector

$p^o \in P$ is said to be an equilibrium price vector if $Z(p^o) \equiv (Z_1(p^o), ..., Z_N(p^o) \leq 0$, where $Z_k(p^o)$ is the excess demand for good k, with $p^o_k = 0$ for k such that $Z_k(p^o) < 0$. That is, p^o is an equilibrium price vector if supply equals demand in all markets (with possible excess supply of free goods).

Theorem 3.0.2: Walras' Law

$\forall p \in P$, $p \cdot Z(p) = \sum_{n=1}^{N} p_n \cdot Z_n(P) = \sum_{i \in H} p \cdot D^i(p) - \sum_{j \in F} p \cdot S^j(p) - p \cdot r = 0$, where $\sum_{i \in H} p \cdot D^i(p)$ is the value of aggregate household expenditure, and $\sum_{j \in F} p \cdot S^j(p) + p \cdot r$ the value of aggregate household income (firm profits plus endowment).

Theorem 3.0.3: Brouwer Fixed Point Theorem

Lef $f(\cdot)$ be a continuous function, $f : P \to P$. Then there is $x^* \in P$ so that $f(x^*) = x^*$.

Theorem 3.0.4: Arrow-Debreu Theorem

Let Walras' law and countinuity be fulfilled, then there is $p^* \in P$ so that p^* is an equilibrium.

3.1 Households

Consider an economy consisting of a unit measure of infinitely-lived households, with measure normalized to 1, uncountable, $\mathscr{H} = [0, 1]$[1]. We assume that households have well-defined preference orderings.

Each household has population L, which grows at the rate $n \geq 0$. Set the initial condition $L(0) = 1$, so that $L(t) = \mathrm{e}^{nt}$. All members are identical, and they supply their one unit of labor inelastically.

Definition 3.1.1: Household Utility Function

Each household i has an instantaneous utility function given by

$$u_i(c_i(t))$$

where $u_i : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave, and $c_i(t)$ is the consumption of household i.

There are two major assumptions of this instantaneous utility function:

- Consumption externalities are ruled out. In other words, household does not derive any utility from the consumption of other households.
- Overall utility is time separable, that is, instantaneous utility at time t is independent of the consumption levels at past or future dates.

The lifetime utility in continuous time can be represented as

$$U(C) = \int_0^\infty \mathrm{e}^{-\rho t} \left[\int_0^{L(t)} u(c_i(t)) \mathrm{d}i \right] \mathrm{d}t$$

where C is the aggregate consumption of the whole household.

Suppose households have asset $\mathscr{A}(t)$, and all assets are invested. We assume that the expenditure is equal to the household income, so that households own the capital

$$C(t) + \dot{\mathscr{A}}(t) = r(t)\mathscr{A}(t) + w(t)L(t)$$

i.e., $\dot{\mathscr{A}}(t) = r(t)\mathscr{A}(t) + w(t)L(t) - C(t)$, where $C(t) = \int_0^{L(t)} c_i(t)\mathrm{d}i$, $r(t)$ is the interest rate, and $w(t)$ is the wage rate. Notice that C is numeraire, all prices are real.

Firstly, we consider a static optimization problem about the optimal allocations within the household:

$$\max_{c_i} \int_0^L u(c_i)\mathrm{d}i$$
$$\text{s.t.} \quad C \geq \int_0^L c_i \mathrm{d}i$$

Obviously, the optimal condition is $c_i = \frac{C}{L}$.

[1] Nothing we do hinges on this assumption.

In this way, we define $c = \frac{C}{L}$, $a = \frac{\mathscr{A}}{L}$, then the household utility becomes

$$\begin{aligned}\int_0^{L(t)} u(c_i(t))\mathrm{d}i &= \int_0^{L(t)} u\left(\frac{C(t)}{L(t)}\right)\mathrm{d}i \\ &= u\left(\frac{C(t)}{L(t)}\right)\int_0^{L(t)} \mathrm{d}i \\ &= u(c(t))\cdot L(t) \\ &= \mathrm{e}^{nt}u(c(t))\end{aligned}$$

and the lifetime utility becomes

$$U(C) = \int_0^{\infty} \mathrm{e}^{-\rho t}\mathrm{e}^{nt}u(c(t))\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-(\rho-n)t}u(c(t))\mathrm{d}t$$

That is the objective function we need. Then we discuss the budget constraint (i.e., the dynamic system) by the identity of expenditure:

$$\begin{aligned}\dot{a}(t) &= \frac{\mathrm{d}a(t)}{\mathrm{d}t} = \frac{\mathrm{d}\frac{\mathscr{A}(t)}{L(t)}}{\mathrm{d}t} \\ &= \frac{\dot{\mathscr{A}}(t)\cdot L(t)}{[L(t)]^2} - \frac{\mathscr{A}(t)\cdot \dot{L}(t)}{[L(t)]^2} \\ &= \frac{r(t)\mathscr{A}(t) + w(t)L(t) - C(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)}\cdot\frac{\mathscr{A}(t)}{L(t)} \\ &= r(t)a(t) + w(t) - c(t) - n\cdot a(t) \\ &= (r(t)-n)\cdot a(t) + w(t) - c(t)\end{aligned}$$

Finally we get the UMP:

$$\begin{aligned}&\max_{c(t)} U(C) = \int_0^{\infty} \mathrm{e}^{-(\rho-n)t}u(c(t))\mathrm{d}t \\ \text{s.t.}\quad &\dot{a}(t) = (r(t)-n)\cdot a(t) + w(t) - c(t)\end{aligned}$$

The current-value Hamiltonian equation of this problem is

$$\mathscr{H} = u(c(t)) + \mu(t)\cdot[(r(t)-n)\cdot a(t) + w(t) - c(t)]$$

The FOCs become

$$\begin{aligned}\frac{\partial\mathscr{H}}{\partial c} &= u'(c(t)) - \mu(t) = 0 \\ \frac{\partial\mathscr{H}}{\partial a} &= \mu(t)\cdot(r(t)-n) = (\rho-n)\mu(t) - \dot{\mu}(t)\end{aligned}$$

By combining the two FOC, we get the Euler equation (see 7.1):

$$\frac{c'}{c} = \frac{r-\rho}{\epsilon}$$

where $\epsilon = -\frac{u''(c)\cdot c}{u'(c)} > 0$.

Then we consider the sufficient conditions of this problem:

1. $\rho > n$, guarantee the math is well-defined.
2. $a(0)$ is given.
3. TVC: $\lim_{t\to\infty} \mathrm{e}^{-(\rho-n)t} \cdot \mu(t) \cdot a(t) = 0$

In this case, the TVC can be solved as (see 7.2):

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s)-n)\mathrm{d}s\right) = 0$$

3.2 Firms

Given the assumptions that there are "no externalities" and that all factors are priced competitively, we can ensure there exists a representative firm. Firm heterogeneity will be relaxed in the endogenous growth model.

Definition 3.2.1: Firm Production Function

The representative firm's production function is given by

$$Y = F(A, K, L)$$

where Y is the output, K is the investment, L is the labor or person and A is a shifter of the production, like efficiency or technology. $F\colon \mathbb{R}^3_+ \to \mathbb{R}_+$ is continuous, strictly increasing and strictly concave in K and L.

We assume that A is public available, nonrival, nonexcludable and satisfies $\dot{A} = 0$. F satisfies several properties:

- Either $K = 0$ or $L = 0$, $Y = F(A, K, L) = 0$.
- Constant return to scale in K and L. That is, F is linear homogeneous.
- Inada conditions: for $j \in \{K, L\}$, $\lim_{j\to 0} F_j = \infty$, $\lim_{j\to\infty} F_j = 0$.

Given A, product price p, wage w and market interest R, the firms choose a set of factors $\{K, L\}$ to maximize its lifetime profit:

$$\max_{K(t),L(t)} \Pi_0 = \int_0^\infty \mathrm{e}^{-rt}\pi_t \mathrm{d}t = \int_0^\infty \mathrm{e}^{-rt}[p(t)Y(t) - R(t)K(t) - w(t)L(t)]\mathrm{d}t$$
$$\text{s.t.} \quad Y(t) \le F(A, K(t), L(t))$$

where F stands for the production possibility frontier.

There is not any difference between π_t and π_{t+1}, so that we can simplify it into a static problem[2]:

$$\max_{K,L} \pi = F(A, K, L) - Rk - wL$$

[2]The price of the output is normalized as 1.

Consider the per capita form, define $y \equiv \frac{Y}{L} = F(A, \frac{K}{L}, 1) \equiv f(k)$, where $k = \frac{K}{L}$, hence the FOCs become

$$\frac{\partial \pi}{\partial K} = F_K - R = f'(k) - R = 0$$
$$\frac{\partial \pi}{\partial L} = \frac{\partial}{\partial L}[f(k)L - RkL - wL] = f(k) - Rk - w = f(k) - k'f(k) - w = 0$$

Thus we have

$$R = f'(k)$$
$$w = f(k) - kf'(k)$$

3.3 Markets Clearing

Theorem 3.3.1: Walras' Law

In a general equilibrium model, one market-clearing constraint is redundant. That is, if each consumer's budget constraint is satisfied and all but one market-clearing conditions hold, then the last market-clearing condition is satisfied automatically.

Consider the markets clearing condition of decentralized markets. Firstly, we normalize the goods price $p(t)$ as 1 for all t. And we assume the capital depreciates at rate $0 < \delta < 1$, and the net interest confronted by households is $r = R - \delta$. The law of capital motion can be written as[3]

$$\dot{k}(t) = i(t) - (n + \delta)k(t)$$

There are three markets which need to satisfy the clearing conditions:

- Commodity market: $c(t) + i(t) = y(t)$.
- Capital market: $a(t) = k(t)$.
- Labor market: $L^s(t) = L^d(t) = L(t)$.

3.4 Competitive Equilibrium

Definition 3.4.1: Competitive Equilibrium

Given $a(0) > 0$, a competitive equilibrium consists of path of allocations for representative households $\{c(t), a(t)\}_{t=0}^{\infty}$, representative firms $\{k(t), L(t)\}_{t=0}^{\infty}$, and price $\{w(t), R(t), r(t)\}_{t=0}^{\infty}$ such that the household allocation solve the UMP, the firm allocation solves the PMP, and prices are such that markets clear.

At the equilibrium point, for representative households, since $R = f'(k)$, the Euler

[3] In this section and the next section, we use the per capita terms.

equation can be rewritten as

$$\frac{\dot{c}}{c} = \frac{1}{\epsilon}(r-\rho) = \frac{1}{\epsilon}(R-\delta-\rho) = \frac{1}{\epsilon}[f'(k) - (\rho+\delta)]$$

For the accumulation of capital, we have

$$\dot{k} = i - (n+\delta)k = y - c - (n+\delta)k = f(k) - (n+\delta)k - c$$

Taking the derivative with respect to c, we obtain

$$\dot{c} = \dot{k}f'(k) - (n+\delta)\dot{k} - \ddot{k}$$

Substituting it into the Euler equation, eventually we get a second-order differential equation with respect to k which uniquely determines a competitive equilibrium:

$$\frac{\dot{k}f'(k) - (n+\delta)\dot{k} - \ddot{k}}{f(k) - (n+\delta)k - \dot{k}} = \frac{1}{\epsilon}[f'(k) - (\rho+\delta)]$$

Thus, the equilibrium is not correlated with price.

3.5 Social Optimum

From the equilibrium equations of decentralized markets, we see that we can only care about the optimal allocations and ignore the prices. There we can think from a social planner's view, whose objective function is the same as the household's problem in decentralized markets, and the budget constraint is the society's feasible resource constraint[4].

The UMP of the social planner's decision is

$$\begin{aligned} \max_{c(t)} U(C) &= \int_0^\infty \mathrm{e}^{-(\rho-n)t} u(c(t))\mathrm{d}t \\ \text{s.t.} \quad \dot{k}(t) &= f(k(t)) - (n+\delta)k(t) - c(t) \end{aligned}$$

given $k(0) > 0$.

The current-value Hamiltonian equation of this problem is

$$\mathscr{H} = u(c) + \mu \cdot [f(k) - (n+\delta)k - c]$$

The FOCs become

$$\begin{aligned} \frac{\partial \mathscr{H}}{\partial c} &= u'(c) - \mu = 0 \\ \frac{\partial \mathscr{H}}{\partial a} &= -\dot{\mu} + (\rho - n)\mu = \mu[f'(k) - (n+\delta)] \end{aligned}$$

and the TVC becomes

$$\lim_{t\to\infty} \mathrm{e}^{-(\rho-n)t}\mu(t)k(t) = \lim_{t\to\infty} k(t)\exp\left(-\int_0^t (f'(k(s) - n - \delta)\mathrm{d}s\right) = 0$$

[4] Since all factors are owned by households, we do not need to consider the fictive "firms" in social planner's decision.

Solving the Euler equation, we obtain

$$\frac{\dot{c}}{c} = \frac{1}{\epsilon}[f'(k) - (\rho + \delta)]$$

Combining with the budget constraint, we have

$$\frac{\dot{k}f'(k) - (n+\delta)\dot{k} - \ddot{k}}{f(k) - (n+\delta)k - \dot{k}} = \frac{1}{\epsilon}[f'(k) - (\rho + \delta)]$$

which is the same as the result in competitive equilibrium.

Theorem 3.5.1: First Welfare Theorem

Suppose we have a competitive equilibrium with allocation $\{c(t), k(t)\}_{t=0}^{\infty}$, then the allocation is Pareto optimal.

Theorem 3.5.2: Second Welfare Theorem

Suppose an allocation $\{c(t), k(t)\}_{t=0}^{\infty}$ is Pareto optimal, then there exists price $\{w(t), R(t), r(t)\}_{t=0}^{\infty}$ that, together with the allocation $\{c(t), k(t)\}_{t=0}^{\infty}$ and $\{a(t)\}_{t=0}^{\infty}$ where $\forall t$, $a(t) = k(t)$, forms a competitive equilibrium.

Corollary 3.5.3

Since the Pareto optimal allocation can be decentralized as a competitive equilibrium, a competitive equilibrium must exist.

As a result of the above analysis, the competitive equilibrium and the Pareto optimum are essentially equivalent. In the following sections, we use the social planner's decision for our discussion.

3.6 Exercises

1. Consider a neoclassical growth model augmented with labor supply decisions. In particular, total population is normalized to 1, and all households have utility function given by

$$\int_0^{\infty} \mathrm{e}^{-\rho t} u\left(c\left(t\right), 1 - l\left(t\right)\right) \mathrm{d}t$$

   where $l\left(t\right) \in (0,1)$ is labor supply. Assume that the production function is $Y(t) = F(K(t), A(t)l(t))$, and $A(t) = \mathrm{e}^{gt}A(0)$.

   (a) Define a competitive equilibrium.

   (b) Set up the current-value Hamiltonian for a planner maximizing the utility of the representative household, and derive the necessary and sufficient conditions for a solution.

   (c) Show that the two problems are equivalent given competitive markets.

Chapter 4

Stability Analysis

Starting from this section, we focus on the solution characterization of the social planner's problem:

$$\max_{c(t)} U(C) = \int_0^{\infty} \mathrm{e}^{-(\rho-n)t} u(c(t))\mathrm{d}t$$
$$\text{s.t.} \quad \dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

given $k(0) > 0$.

The dynamic system is determined by the following two equations:

$$\frac{\dot{c}}{c} = \frac{1}{\epsilon}[f'(k) - (\rho+\delta)]$$
$$\dot{k} = f(k) - (n+\delta)k - c$$

4.1 Steady State

Definition 4.1.1: Steady State

A steady state is defined as an equilibrium path in which the growth rate of all endogenous variable are zero (mathematically, reaches a fixed point).

On steady state, since $\dot{c} = 0$ and $\dot{k} = 0$, we have

$$f'(k^*) = \rho + \delta$$
$$c^* = f(k^*) - (n+\delta)k^*$$

Thus, the net interest $r^* = R^* - \delta = f'(k^*) - \delta = \rho$, and the saving rate $s^* \equiv \frac{y^*-c^*}{y^*} = \frac{(n+\delta)k^*}{f(k^*)}$.

In the Solow model, the golden rule is defined as the steady state capital stock that maximizes the steady state consumption:

$$\max_{k} c = f(k) - (n+\delta)k$$

where FOC gives

$$f'(k^G) = n + \delta$$

Since $\rho > n$, we have $f'(k^*) > f'(k^G)$ and $k^* < k^G$. We call k^* the modified golden rule.

Then we discuss the comparative statics. Consider the technological changes, we suppose that $f(k) = A\tilde{f}(k)$, i.e. $\tilde{f}(k^*) = \frac{\rho+\delta}{A}$. For the two exogenous variables c^*, k^* and four endogenous variables A, ρ, n, δ, we have several conclusions below.

For k^*, it is easy to know that

$$\begin{aligned} \frac{\mathrm{d}k^*}{\mathrm{d}A} &> 0 \qquad & \frac{\mathrm{d}k^*}{\mathrm{d}\rho} &< 0 \\ \frac{\mathrm{d}k^*}{\mathrm{d}n} &= 0 \qquad & \frac{\mathrm{d}k^*}{\mathrm{d}\delta} &< 0 \end{aligned}$$

For c^*, we also have

$$\begin{aligned} \frac{\mathrm{d}c^*}{\mathrm{d}A} &= \frac{\partial c^*}{\partial A} + \frac{\partial c^*}{\partial k^*}\frac{\mathrm{d}k^*}{\mathrm{d}A} \\ &= \tilde{f}(k^*) + [A\tilde{f}(k^*) - (n+\delta)]\frac{\mathrm{d}k^*}{\mathrm{d}A} \\ &= \tilde{f}(k^*) + (\rho - n)\frac{\mathrm{d}k^*}{\mathrm{d}A} > 0 \\ \frac{\mathrm{d}c^*}{\mathrm{d}\rho} &= \frac{\partial c^*}{\partial k^*}\frac{\mathrm{d}k^*}{\mathrm{d}\rho} = (\rho - n)\frac{\mathrm{d}k^*}{\mathrm{d}\rho} < 0 \\ \frac{\mathrm{d}c^*}{\mathrm{d}n} &= \frac{\partial c^*}{\partial k^*}\frac{\mathrm{d}k^*}{\mathrm{d}n} + \frac{\partial c^*}{\partial n} \\ &= \frac{\partial c^*}{\partial n} = -k^* < 0 \\ \frac{\mathrm{d}c^*}{\mathrm{d}\delta} &= \frac{\partial c^*}{\partial k^*}\frac{\mathrm{d}k^*}{\mathrm{d}\delta} + \frac{\partial c^*}{\partial \delta} \\ &= (\rho - n)\frac{\mathrm{d}k^*}{\mathrm{d}\delta} - k^* < 0 \end{aligned}$$

4.2 Local Stability

According to Lyapunov's second method, we linearize this non-linear dynamic system using CRRA utility function (see 7.3), then formalize the linear system, we have

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} \rho - n & -1 \\ \frac{1}{\epsilon}c^* f''(k^*) & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix}$$

The local stability depends on the eigenvalue ζ of the system, which is defined as

$$\left| \begin{bmatrix} \rho - n & -1 \\ \frac{1}{\epsilon}c^* f''(k^*) & 0 \end{bmatrix} - \zeta I \right| = \begin{vmatrix} \rho - n - \zeta & -1 \\ \frac{1}{\epsilon}c^* f''(k^*) & -\zeta \end{vmatrix} = 0$$

That is

$$\zeta^2 + \beta_1\zeta + \beta_2 = 0$$

where $\beta_1 = -(\rho - n) < 0$, $\beta_2 = \frac{c^* f''(k^*)}{\epsilon} < 0$.

Since $\beta_1 - 4\beta_2 > 0$ and $\beta_2 < 0$, there are two real eigenvalues $\lambda_1 < 0$, $\lambda_2 > 0$. Thus, there exists a saddle path: λ_1 refers to the stable path, and λ_2 refers to the unstable path.

In addition, we consider the same problem in discrete time. We use the log-linearization method to expand linear series in terms of the rate of change:

$$\hat{x} \equiv \frac{x - x^*}{x^*} \approx \ln\left(1 + \frac{x - x^*}{x^*}\right) = \ln\left(\frac{x}{x^*}\right) = \ln x - \ln x^*$$

4.3 Global Stability

We already know that there exists a fixed point (k^*, c^*), and it is locally stable with a saddle path. Then we analyze the dynamic system in a (k, c) phase diagram to study the global stability.

Initially, we set $\dot{k} = 0$, then $c = f(k) - (n+\delta)k$. Since $(n+\delta)k$ is linear, and $f(k)$ is strictly concave, satisfying the Inada condition, we know that the curve $c|_{\dot{k}=0}$ is strictly concave. Consider some special point, $c = 0$ iff $k = 0$ or $f(\bar{k}) = (n+\delta)\bar{k}$, and c is maximized if $k = k^G$.

In the horizontal direction, if $\dot{k} < 0$, i.e., a leftward velocity, then $c > f(k) - (n+\delta)k$. And if there is a rightward velocity, we know that $c < f(k) - (n+\delta)k$.

Secondly, we set $\dot{c} = 0$ gives $k = k^* < k^G$. In the vertical direction, since $f'(k)$ is strictly decreasing, if $\dot{c} > 0$, i.e., an upward velocity, then $f'(k) > \rho + \delta$, that is $k < k^*$. If there is a downward velocity, we know that $k > k^*$.

Combining both the two dimensions, it can be illustrated as a phase diagram:

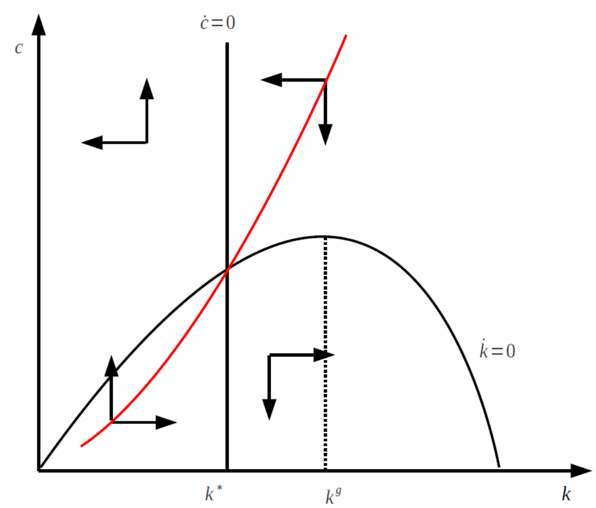


Figure 4.1: Illustration: $\dot{c} = 0$ and $\dot{k} = 0$

Actually, the path starting with $k(0), c(0)$ and converging to (k^*, c^*) is unique by the sufficient conditions of the optimization problem (as shown by the red line in the phase diagram).

4.4 Exercises

1. Consider the following investment problem:

$$\max_{I(t)} \int_0^\infty \mathrm{e}^{-\rho t} \left[f(k(t) - I(t) - \frac{\chi}{2} \left(\frac{I(t)^2}{k(t)} \right) \right] \mathrm{d}t$$

s.t.

$$\dot{k}(t) = \dot{I}(t)$$

given $k(0)$. The interest rate $r > 0$, the capital adjustment cost $\chi > 0$, and the production function is strictly positive and strictly concave.

   (a) Determine the Euler equation.

   (b) Solve for the steady state.

   (c) Show that the steady state is a saddle point locally.

   (d) Draw the phase diagram.

   (e) If the capital accumulation equation turns into $\dot{k}(t) = I(t) - \delta k(t)$, where $\delta \in (0, 1)$, resolve the questions above.

2. Consider the following problem:

$$\max_{c(t), l(t)} \int_0^\infty \mathrm{e}^{-\rho t} \ln(c) \mathrm{d}t$$

s.t.

$$\dot{k} = y - c - \delta k$$
$$\dot{h} = Bh(1 - l)$$

given initial human capital h_0 and initial physical capital k_0.
Output y_t is defined by

$$y \equiv Ak^\alpha (lh)^\beta$$

We also impose $0 \leq l_t \leq 1$. The discount factor $\rho > 0$, the rate of depreciation of physical capital $\delta > 0$, the constants $A > 0$ and $B > 0$, and the production function parameters $\alpha \in (0, 1)$ and $\beta \in (0, 1)$.

   (a) Determine the Euler equation.

   (b) From now on, we assume that $B = 0$. Show that $l = 1$. Draw the phase diagram in the (k, c) plane.

   (c) Show on the diagram that the steady state of the system is a saddle point.

   (d) Derive the Jacobian matrix of the system.

   (e) Show that the steady state of the system is a saddle point.

3. Consider the following problem:

$$\max_{c(t)} \int_0^\infty \mathrm{e}^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \mathrm{d}t$$

s.t.

$$\dot{k}(t) = k(t)^\alpha - c(t) - \delta k(t)$$

given $k(0)$, where $\alpha \in (0,1)$, $\delta \in (0,1)$, $\rho \in (0,1)$, and $\sigma > 0$. Impose that $\rho + \delta < 1$.

(a) Determine the Euler equation.

(b) Suppose $\alpha = 1$ and $\sigma = 1$. Show that the system describing the optimal functions $\{k(t), c(t)\}$ reduces to a linear, homogeneous system of first-order differential equations. Show that the system is unstable by computing the eigenvalues.

(c) Suppose $\alpha < 1$ and $\sigma > 0$. Show that the system describing the optimal functions $\{k(t), c(t)\}$ reduces to a nonlinear system of first-order differential equations. Use a phase diagram to show that the steady state of the system is a saddle point.

Chapter 5

Numerical Methods

5.1 Model Determination

We continue to focus on the social planner's problem:

$$\max_{c(t)} U(C) = \int_0^\infty \mathrm{e}^{-(\rho-n)t} u(c(t))\mathrm{d}t$$
$$\text{s.t.} \quad \dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

given $k(0) > 0$.

The dynamic system is determined by the following two equations:

$$\frac{\dot{c}}{c} = \frac{1}{\epsilon}[f'(k) - (\rho+\delta)]$$
$$\dot{k} = f(k) - (n+\delta)k - c$$

In order to find the numerical solution of this problem, we need to set the specific model form. For simplicity (literally, the analytical solution exists), we set the utility function $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, therefore $\epsilon = \theta$, and the production function $y = f(k) = Ak^\alpha$.

Then we determine the numbers of all the exogenous parameters, then use algorithms to calibrate them.

For example, we determine the scale parameter $A = 1$, the capital share $\alpha = \frac{1}{3}$, and the depreciation rate $\delta = 0.05$. The discount rate ρ can be indirectly represented as r^*, which is the steady state real interest rate. In this case, we determine $\rho = 0.05$. Suppose that there is no population growth ($n = 0$) for simplicity again. Finally, we set $\theta = \alpha$ for closed-form solution.

The analytical solution can be represented as (see 7.4)

$$c = \frac{\rho + (1-\alpha)\delta}{\alpha} k$$

In the next section, we introduce a numerical algorithm called the shooting method in order to transform the BVP of second-order differential equations into an IVP. We illustrate the result by single forward shooting:

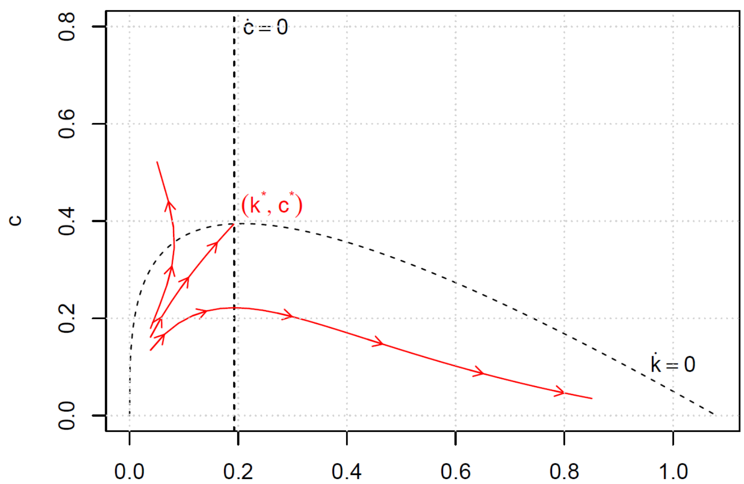


Figure 5.1: Illustration: Numerical Solution

5.2 Single Forward Shooting Method: Principle

Starting from this section, we focus on a similar problem but in discrete time (continuous time is harder to compute):

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$
$$\text{s.t.} \quad c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha}$$

given $k_0 > 0$.

The Euler equation can be solved as

$$\frac{c_{t+1}}{c_t} = \beta \left[\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right]$$

Combining with the budget constraint, we obtain a second-order difference equation:

$$\frac{k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}}{k_t^{\alpha} + (1-\delta)k_t - k_{t+1}} = \beta \left[\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right]$$

with unknowns k_{t+2}, k_{t+1} and k_t.

It is a standard boundary value problem:

$$\begin{aligned} k_{t+2} &= f(t, k_t, k_{t+1}) \\ &= k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - \beta \left[\alpha k_{t+1}^{\alpha-1} + (1-\delta)\right] \left[k_t^{\alpha} + (1-\delta)k_t - k_{t+1}\right] \end{aligned}$$

with $k_0 \leq k_t \leq k^*$.

If k_1 is known, we can iterate the difference equation to get $k_2, k_3, \ldots$. Hence, the BVP turns into an initial value problem. We define the solution of the difference equation as $k_t = g(t, k_1)$, then $\lim_{t\to\infty} g(t, k_1) = k^*$. The IVP is to find k_1 to solve the function

$$\lim_{t\to\infty} g(t, k_1) - k^* = 0$$

5.3 Single Forward Shooting Method: Programming

First, we need to guess c_0, then iterate for T (large enough) periods. The Python code is as follows:

```
import numpy as np
def consumption_path(c0, n=100): # set T=100
    '''shooting for consumption path'''
    k = np.zeros(n)
    c = np.zeros(n)
    k[0] = k0
    c[0] = c0
    for i in range(n-1):
        k[i+1] = k[i]**alpha-c[i]
        if k[i+1] <= 0: # no free lunch
            k[i+1] = 0
            break
        c[i+1] = c[i]*beta*(alpha*k[i+1]**(alpha-1))
    return k, c
k1, c1 = consumption_path (c0 = 0.0044) # test
```

Then we compare (k_T, c_T) with the target (k^*, c^*) and adjust it. In this example, we use the bisection method to construct our code:

```
def bisection(c0_min=1e-6, c0_max=c_max, tol=1e-16, max_iter=1000):
    '''bisection method'''
    count = 0
    while count < max_iter and c0_max-c0_min > tol:
        count = count+1
        c0 = (c0_min+c0_max)/2
        k, c = consumption_path(c0)
        if k[-1] < kss:
            c0_max = c0
        else:
            c0_min = c0
    if c0_max-c0_min > tol:
        print("Failed to converge!")
    else:
        print(f"\nConverged in {count} iterations.")
        print("c0 =", c0)
    return c0
import timeit
start_time = timeit.default_timer() # test
c0 = bisection()
print("Duration:", timeit.default_timer() - start_time)
```

When $\delta = 1$, this model has analytical solutions. The optimal decision path can be solved as $c_t = (1 - \alpha\beta)k_t^\alpha$, $k_{t+1} = \alpha\beta k_t^\alpha$. And the steady state $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$, $c^* = k^\alpha - k$. We can define the analytical solution in Python:

```
alpha = 0.9 # to have slower transition speed
beta = 0.95
kss = (alpha*beta)**(1/(1-alpha)) # steady state of k
k0 = kss/10 # initial k at time 0

def css(k):
    '''c locus'''
    return (k**alpha-k)

k_G = alpha**(1/(1-alpha)) # golden rule of capital
c_max = css(k_G)

def c_analytical_compute(k):
    '''calculate analytical solution'''
    return (1-alpha*beta)*k**alpha
```

Comparing with the numerical solution by the shooting method, we get the phase diagram below:

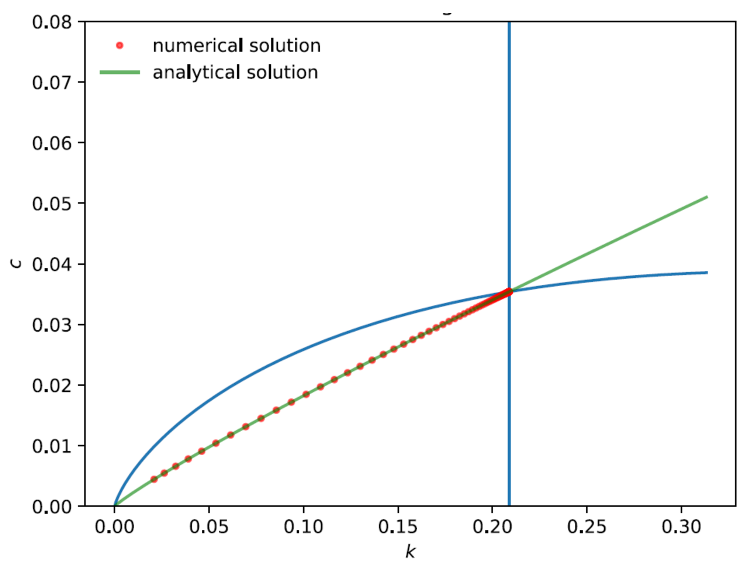


Figure 5.2: Comparison between Two Solutions

5.4 Exercises

1. Consider the following social planner's problem. There is no technical change and no population growth. The production function for final output per capita is $y = k^{\alpha}$.

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \frac{(c_t - \bar{c})^{1-\gamma}}{1-\gamma}$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t \leq k_t^{\alpha}$$

with $k_0 > 0$ given. $\bar{c} \geq 0$ is called the subsistence consumption needs. Assume that the economy starts above the subsistence requirement, i.e., $k_0^{\alpha} > \bar{c}$.

(a) Derive the Euler equation and the steady states of this model.

(b) Let $\alpha = 0.3$, $\beta = 0.95$, $\delta = 0.1$, $\gamma = 1.5$, $k_0 = k^*/5$. k^* denotes the steady state of capital. Use single forward shooting to solve the transitional dynamics of the resulting dynamic system for the following different value of $\bar{c}$. As an illustration of result, first plot the policy function $c = f(k)$ (i.e., the stable saddle path) and the steady state curve of $\dot{c} = 0$ and $k = 0$ in a (c, k) locus, then plot the time path of the saving rate $s \equiv \frac{y-c}{y}$.

i. Suppose $\bar{c} = 0$.

ii. Suppose $\bar{c} = 0.5$.

iii. Compare the time path of the saving rate between two cases, explain why they are different.

Chapter 6

Exogenous Growth

6.1 Technology

In the previous chapters, we assumed that technology is constant, so the only engine of growth is population. Now we come to the case where exogenous technological progress exists. There are three types of production function that include technology:

- Hicks neutral: $Y = AF(K.L) = F(AK, AL)$.
- Solow neutral: $Y = F(AK, L)$, as known as capital augmenting.
- Harrod neutral: $Y = F(K, AL)$, as known as labor augmenting.

For Cobb-Douglas production function, they are mathematically the same:

$$\begin{aligned} Y_{Harrod} &= K^\alpha (AL)^\beta = A^\beta K^\alpha L^\beta \equiv \tilde{A} K^\alpha L^\beta \\ Y_{Solow} &= (AK)^\alpha L^\beta = A^\alpha K^\alpha L^\beta \equiv \tilde{A} K^\alpha L^\beta \end{aligned}$$

where $\alpha + \beta = 1$.

In this section, we assume that technology grows at a constant rate $g > 0$, that is $A(t) = A(0)\mathrm{e}^{gt}$, and utility $u(c) = \frac{c^{1-\theta}}{1-\theta}$, and production $Y = F(K, AL)$ (Harrod neutral). This new model also satisfies the first welfare theorem, so that we only discuss the social planner's problem:

$$\max_{C(t)} \int_0^\infty \mathrm{e}^{-(\rho-n)t} \frac{\left(\frac{C(t)}{L(t)}\right)^{1-\theta}}{1-\theta} \mathrm{d}t$$

s.t.

$$\dot{K}(t) = Y(t) - \delta K(t) - C(t)$$

For the benchmark model, only the population grows at a constant exogenous rate, so that the model in per capita form has fixed point (i.e., $\dot{c} = x(c)$ instead $x(c,t)$, so do $\dot{k}$). We normalize the new model into a per effective unit form: Let $c = \frac{C}{AL}$, $k = \frac{K}{AL}$ and $y = \frac{Y}{AL}$, with CRS assumption, we have

$$\begin{aligned}
\frac{\dot{K}}{K} &= \frac{Y}{K} - \delta - \frac{C}{K} \\
&\xlongequal{\text{Euler's Theorem}} \frac{Y_K K + Y_{AL} AL}{K} - \delta - \frac{c}{k} \\
&= F_K(k,1) + \frac{F_{AL}(k,1)}{k} - \delta - \frac{c}{k} \\
&= \frac{f(k)}{k} - \delta - \frac{c}{k}
\end{aligned}$$

Then the constrain condition becomes

$$\begin{aligned}
\dot{k} &= \left(\frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}\right) k \\
&= \left(\frac{y}{k} - \delta - \frac{c}{k} - n - g\right) k \\
&= f(k) - c - (n + \delta + g)k
\end{aligned}$$

Similarly, the objective function becomes

$$\begin{aligned}
U &= \int_0^\infty \mathrm{e}^{-(\rho-n)t} \frac{(A_0 \mathrm{e}^{gt} c)^{1-\theta}}{1-\theta} \mathrm{d}t \\
&= A_0^{1-\theta} \int_0^\infty \mathrm{e}^{-[\rho-(1-\theta)g-n]t} \frac{c^{1-\theta}}{1-\theta} \mathrm{d}t
\end{aligned}$$

Hence, the problem becomes

$$\max_c \int_0^\infty \mathrm{e}^{-[\rho-(1-\theta)g-n]t} \frac{c^{1-\theta}}{1-\theta} \mathrm{d}t$$

s.t.

$$\dot{k} = f(k) - c - (n + \delta + g)k$$

given $k(0) > 0$.

6.2 Solow Growth Model

Economic growth, according to Kaldor (1961), exhibits these characteristics (that we call "Kaldor's stylized facts"):

- The growth rate of per capita output g_y is constant.
- Per capita output Y/L and per capita capital K/L grow at the same rate.
- The wage rate w is growing continuously.
- The (real) interest rate r is constant (so do R due to the constant δ).
- The capital-output ratios K/Y is constant.
- Factor income shares $\frac{wL}{Y}$ and $\frac{RK}{Y}$ are both constant (that's why we use Cobb-Douglas).

During steady state, the Harrod neutral production derives $w = AF_L$ and $R = F_K$. Therefore, only this form of production function fits the Kaldor's facts. Distinguish from benchmark model, the Solow model does not include household decisions, but uses the exogenous savings rate assumption $s \equiv \frac{Y-C}{Y}$ as an alternative.

The model can be simply represented as

$$\begin{aligned} Y &= F(K, AL) \\ \dot{K} &= sY - \delta K \end{aligned}$$

The dynamic system can also be written in per effective unit form:

$$\dot{k} = sf(k) - (\delta + n + g)k$$

Suppose $\dot{k} = 0$, the steady state becomes

$$sf(k^*) = (\delta + n + g)k^*$$

which means that the new investment equals the diluted capital[1].

It is easy to show the comparative statics:

$$\begin{array}{ll} \dfrac{\partial k^*}{\partial \delta} < 0 & \dfrac{\partial k^*}{\partial n} < 0 \\ \dfrac{\partial k^*}{\partial g} < 0 & \dfrac{\partial k^*}{\partial s} > 0 \end{array}$$

On the balanced growth path, since k^* is constant, $y^* = f(k^*)$ is also a constant. Corresponding to the Kaldor's facts, we have:

- $\frac{Y}{L} = Ay^*$ grows at a constant rate g, so do $\frac{K}{L} = Ak^*$.
- $w = A[f(k^8) - k^* f'(k^*)]$ grows at a constant rate g.
- $R = f'(k^*)$ is constant.
- $\frac{K}{Y} = \frac{k^*}{y^*} = \frac{s}{\delta+n+g}$ is constant.
- $\frac{wL}{Y} = \frac{AL[f(k^8)-k^* f'(k^*)]}{Y} = \frac{f(k^*)-k^* f'(k^*)}{y^*}$ is constant.
- $\frac{RK}{Y} = f'(k^*)\frac{s}{\delta+n+g}$ is constant.

Decomposing growth rate of the output per capita $\kappa \equiv \frac{Y}{L}$ into two components:

$$\begin{aligned} \frac{\dot{\kappa}}{\kappa} &= \frac{\mathrm{d}\ln(Af(k))}{\mathrm{d}t} \\ &= \frac{\dot{A}}{A} + \frac{f'(k)}{f(k)}\frac{\mathrm{d}k}{\mathrm{d}t} \\ &= g + \frac{f'(k)}{f(k)}[sf(k) - (\delta + n + g)k] \\ &\equiv \gamma_A + \gamma_k \end{aligned}$$

[1] The existence of k^* can be ensured by the Inada conditions.

where $\gamma_A = g$ stands for the part of long-term technological changes, and γ_k stands for the part of short-term capital accumulation.

Suppose the production function is Cobb-Douglas $f = k^\alpha$, with the accumulation of capital, γ_k is decreased:

$$
\begin{aligned}
\frac{\partial \gamma_k}{\partial k} &= \frac{\partial \left[s' f(k) - (\delta + n + g) k \frac{f'(k)}{f(k)} \right]}{\partial k} \\
&= s f''(k) - (\delta + n + g)\alpha < 0
\end{aligned}
$$

In the long term, γ_k will converge to zero, so only γ_A exists:

$$
\lim_{t \to \infty} \gamma_k = \lim_{k \to k^*} \gamma_k = \frac{f'(k^*)}{f(k^*)} [s f(k^*) - (\delta + n + g) k^*] = 0
$$

Therefore, technological changes is the only engine of long-run growth, while capital accumulation only affect the rate of nonstationary growth in short-run.

6.3 Empirical Evidence

We already introduce the basic model of Solow growth, and empirically, Mankiw et al. (1992) examines whether the Solow growth model is consistent with the international variation in the standard of living.

When production function $Y = F(K, AL) = K^\alpha (AL)^{1-\alpha}$,The steady state of Solow model implicates that

$$
k^* = \left(\frac{s_k}{n + g + \delta} \right)^{\frac{1}{1-\alpha}}
$$

Combining with the production function and taking logs, we find that steadystate income per capita is

$$
\ln \left[\frac{Y(t)}{L(t)} \right] = \frac{\alpha}{1-\alpha} \ln s_k - \frac{\alpha}{1-\alpha} \ln(n + g + \delta) + \ln A_0 + gt
$$

This decomposing equation predicts that real income is higher in countries with higher saving rates and lower in countries with higher values of $(n + g + \delta)$, and this paper manages to investigate whether it is true.

We assume that g and δ are constant across countries. In contrast, the $A(0)$ term reflects not just technology, but resource endowments, climate, institutions, and so on; it may therefore differ across countries. We assume that

$$
\ln A(0) = a + \epsilon
$$

where a is a constant and ϵ is a country-specific shock. Thus, log income per capita at time 0 is

$$
\ln \left(\frac{Y}{L} \right) = a + \frac{\alpha}{1-\alpha} \ln s - \frac{\alpha}{1-\alpha} \ln(n + g + \delta) + \epsilon
$$

Furthermore, we assume that $s, n \perp \epsilon$. This implies that we can estimate this equation with OLS.

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	5.48	5.36	7.97
	(1.59)	(1.55)	(2.48)
ln(I/GDP)	1.42	1.31	0.50
	(0.14)	(0.17)	(0.43)
$\ln(n + g + \delta)$	−1.97	−2.01	−0.76
	(0.56)	(0.53)	(0.84)
$\overline{R}^2$	0.59	0.59	0.01
s.e.e.	0.69	0.61	0.38
Restricted regression:			
CONSTANT	6.87	7.10	8.62
	(0.12)	(0.15)	(0.53)
$\ln(I/GDP) - \ln(n + g + \delta)$	1.48	1.43	0.56
	(0.12)	(0.14)	(0.36)
$\overline{R}^2$	0.59	0.59	0.06
s.e.e.	0.69	0.61	0.37
Test of restriction:			
p-value	0.38	0.26	0.79
Implied α	0.60	0.59	0.36
	(0.02)	(0.02)	(0.15)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

Figure 6.1: Estimation of the textbook Solow model

The result tells that the coefficients on $\ln s$ and $\ln(n+g+\delta)$ are are equal in magnitude and opposite in sign is not rejected in any of the samples. More importantly, differences in saving and population growth account for a large fraction of the cross-country variation in income per capita. Nonetheless, the model is not completely successful, the data strongly contradict the prediction that $\alpha = \frac{1}{3}$.

Including human capital can potentially alter either the theoretical modeling or the empirical analysis of economic growth. We consider a augmented Solow model: the production function becomes $Y = K^\alpha H^\beta (AL)^{1-\alpha-\beta}$, and the dynamic system becomes

$$\dot{k} = s_k k^\alpha h^\beta - (n+g+\delta)k$$
$$\dot{h} = s_h k^\alpha h^\beta - (n+g+\delta)h$$

Thus, the steady state is

$$k^* = \left(\frac{s_k^{1-\beta} s_h^\beta}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}}$$
$$h^* = \left(\frac{s_h^\alpha s_h^{1-\alpha}}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}}$$

and the OLS model becomes

$$\ln\left(\frac{Y}{L}\right) = a + \frac{\alpha}{1-\alpha-\beta}\ln s_k + \frac{\beta}{1-\alpha-\beta}\ln s_h - \frac{\alpha+\beta}{1-\alpha-\beta}\ln(n+g+\delta) + \epsilon$$

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\bar{R}^2$	0.78	0.77	0.24
s.e.e.	0.51	0.45	0.33
Restricted regression:			
CONSTANT	7.86	7.97	8.71
	(0.14)	(0.15)	(0.47)
$\ln(I/GDP) - \ln(n + g + \delta)$	0.73	0.71	0.29
	(0.12)	(0.14)	(0.33)
$\ln(SCHOOL) - \ln(n + g + \delta)$	0.67	0.74	0.76
	(0.07)	(0.09)	(0.28)
$\bar{R}^2$	0.78	0.77	0.28
s.e.e.	0.51	0.45	0.32
Test of restriction:			
p-value	0.41	0.89	0.97
Implied α	0.31	0.29	0.14
	(0.04)	(0.05)	(0.15)
Implied β	0.28	0.30	0.37
	(0.03)	(0.04)	(0.12)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Figure 6.2: Estimation of the augmented Solow model

The results in the table above strongly support the augmented Solow model. For non-oil and intermediate samples, the implied α and β in the restricted regression are about one third and highly significant. The estimates for the OECD alone are less precise. This proves that adding human capital to the Solow model do improve its performance.

6.4 Growth Accounting

In Solow model, there is only one parameter which is unobservable – technology, or TFP. We will introduce two basic method to calculate the growth of technology from the statistical data.

The first one is called Solow residual method. Consider a general Cobb-Douglas production function $Y = F(A, K, L) = AK^{\alpha}L^{1-\alpha}$, the per capita form is $y = \frac{Y}{L} = \frac{AK^{\alpha}}{L^{\alpha}} = Ak^{\alpha}$. Thus, the growth rate can be written as $\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}$, the per capita form is $\frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \alpha\frac{\dot{k}}{k}$. So the growth of technology is

$$\frac{\dot{A}}{A} = \frac{\dot{y}}{y} - \alpha\frac{\dot{k}}{k}$$

But the capital always has measuring errors, so on the other side, assuming CRS and competitive markets, we have $Y = RK + wL$, the dual method means

$$\frac{\dot{A}}{A} = \alpha\frac{\dot{R}}{R} + (1-\alpha)\frac{\dot{w}}{w}$$

In this method, the market interest R and the wage rate w both are market-determined and therefore can be accurately measured.

By Kaldor's facts, the distribution of income between capital and labor remain roughly constant. The figure below shows the evolution of the shares of capital and labor in the U.S. national income:

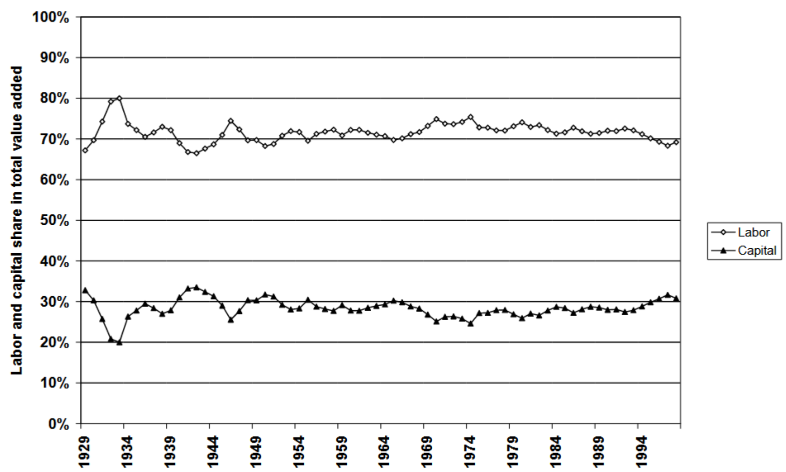


Figure 6.3: The Evolution of the Factor Share in the U.S. National Income

Suppose the labor share $1-\alpha = \frac{2}{3}$, we use the residual method into the U.S. data[2]:

Period	$\frac{\dot{Y}}{Y}$	$\frac{\dot{K}}{K}$	$\frac{\dot{H}}{H}$	$\frac{\dot{A}}{A}$
1950-2014	3.11	3.03	1.91	0.83
1950-1973	3.95	3.82	2.42	1.07
1973-1993	2.81	2.81	2.10	0.48
1993-2007	3.19	2.92	1.64	1.13
2007-2014	1.01	1.27	0.28	0.40

Table 6.1: The U.S. Growth Accounting (1950-2014)

In the long run, factor accumulation in the U.S. has shown a downward trend over time, so only technological change is the engine of growth.

Unlike the U.S. which has been in a steady state, Young (1995) carried out an accounting of East Asian countries' post-WWII growth miracle using residual method:

It can be seen that the growth of Hong Kong, South Korea, and Taiwan, all stem from technological changes. However, Singapore mainly relies on factor accumulation. This does not align with the conclusions of the Solow model.

[2]The labor L is measured by the human capital H, that is, the educational attainment.

Countries/Regions	$\frac{\dot{Y}}{Y}$	$\frac{\dot{K}}{K}$	$\frac{\dot{H}}{H}$	$\frac{\dot{A}}{A}$	$1-\alpha$
Hong Kong (66-91)	7.3	8.0	3.2	2.3	0.63
South Korea (66-90)	10.3	13.7	6.4	1.7	0.70
Singapore (66-90)	8.7	11.5	5.7	0.2	0.51
Taiwan (66-90)	9.4	12.3	4.9	2.6	0.74

Table 6.2: The Growth Accounting in East Asia (1966-1991)

Hsieh (2002) re-examined the "East Asian growth miracle" using both the primal (residual) and dual methods:

		Annual growth rate of:			
Real interest rate used	Labor share	Rental price of capital	Wages	Dual TFP	Primal TFP
Korea					
Curb market loan rate (1966–1990)	0.703	−3.95	4.38	1.91	1.70
Deposit rate (1966–1990)	0.703	−3.41	4.38	2.07	1.70
Discount rate (1966–1990)	0.703	−4.91	4.38	1.62	1.70
Singapore (actual real interest rate)					
Return on equity (1971–1990)	0.511	−0.20	3.17	1.52	−0.69
Average lending rate (1968–1990)	0.511	1.64	2.67	2.16	−0.22
E-P ratio (1973–1990)	0.511	−0.50	3.63	1.61	−0.66
Singapore (constant real interest rate)					
(1971–1990)	0.511	0.34	3.17	1.78	−0.69
(1968–1990)	0.511	0.60	2.67	1.65	−0.22
(1972–1990)	0.511	0.08	3.63	1.89	−0.66
Hong Kong					
Prime lending rate (1966–1991)	0.628	−1.13	4.05	2.12	2.30
Call money rate (1966–1991)	0.628	−1.53	4.05	1.98	2.30
E-P ratio (1973–1991)	0.616	0.96	4.14	2.92	2.18
Taiwan					
Curb loan rate (1966–1990)	0.739	−0.36	5.26	3.79	2.60
One-year deposit rate (1966–1990)	0.739	−0.07	5.26	3.87	2.60
Secured loan rate (1966–1990)	0.739	−2.01	5.26	3.36	2.60
Three-month treasury bill rate (1975–1990)	0.746	−2.07	5.79	3.79	2.70

Figure 6.4: Dual TFP Growth in East Asia (1966-1991)

This paper shows that TFP measured by the primal method has big errors. But the dual method gives more accurate results. This also proves the Solow model's success in explaining East Asian countries' economic growth.

Unlike these small countries or regions, China has a wholly different method of growth. During the reform period (1978-1998), China gave a outstanding growth performance, with breathtaking magnitude (8% per year) and inordinate endurance. But due to the CPC's secret politics, the official statistics of China are doubted for a long time.

Young (2003) studied whether China's growth performance is truly extraordinary, and whether China's official data is reliable. Firstly accepting all the official numbers, with internal systematic adjustments. But the standard growth accounting only involved non-agricultural sector, since the agricultural sector is predominantly self-employed, which is not satisfied the assumption of Solow model.

The table below shows the official data is internally consistent and generally reliable, but this paper found that the growth rate is largely reduced: non-agricultural labor productivity fell by 2.6%, and the total factor productivity fell by 1.4%. The adjusted growth accounting results show that China's growth performance is similar to the world average.

Young's result argued the "China's miracle". However, Only focusing on the non-

	Official	Adjusted
Aggregate:		
Output per capita	7.8	6.1
Output per worker	6.9	5.2
Nonagricultural:		
Output per worker	6.1	3.6
Output per effective worker	5.0	2.6
Output per unit of capital	1.4	.4
Total factor productivity	3.0	1.4

Figure 6.5: China's Growth Accounting (1978-1998)

agricultural sector growth clearly does not align with the fact.

Zhu (2012) re-argured Young's conclusion, it had excessively underestimated China's growth. And this paper decomposed the contributions of different factors to per capita GDP growth to explain which is the real engine.

	Average annual growth rates (%)				
Period	*GDP per capita*	*Labor partication rate*	*Capital/output ratio*	*Average human capital*	*TFP*
1952–1978	2.97	0.11	3.45	1.55	−1.07
1978–2007	8.12	0.57	0.04	1.18	3.16
	Contributions to per capita GDP growth				
Period	*GDP per capita*	*Labor partication rate*	*Capital/output ratio*	*Average human capital*	*TFP*
1952–1978	100	3.63	116.15	52.25	−72.03
1978–2007	100	7.05	0.51	14.55	77.89

Figure 6.6: Decomposing China's Growth (1952-2007)

According to the results, we find that before the reform, the accumulation of physical capital was the main driving force for growth, and at the same time, human capital accumulated greatly due to literacy campaigns. After the 1980s, technological change became the main engine of growth, which benefited from the earlier accumulation of human capital.

	Average annual total factor productivity growth (%)			
		Nonagricultural sector		
Period	*Agriculture*	*Nonstate*	*State*	*Aggregate*
1978–2007	**4.01**	**3.91**	**1.68**	**3.61**
1978–1988	2.79	5.87	−0.36	3.83
1988–1998	5.10	2.17	0.27	2.45
1998–2007	4.13	3.67	5.50	4.68

Figure 6.7: TFP Growth by Sector

This table shows that the TFP of the agricultural sector has been growing steadily, and in some periods it even became the main part of growth. The results of this paper indicate that growth accounting that only calculates the non-agricultural sector is unreliable.

Chapter 7

Appendices

7.1 The solution of the Hamiltonian equation in 3.1

Rewrite the two FOCs:

$$u'(c(t)) = \mu(t)$$
$$\mu(t) \cdot (r(t) - n) = (\rho - n)\mu(t) - \dot{\mu}(t)$$

Rearranging the second equation, we have

$$\frac{\dot{\mu}(t)}{\mu(t)} = (\rho - r)$$

Combining with the first equation:

$$\text{LHS} = \frac{u''(c) \cdot \dot{c}}{u'} = \frac{\dot{c}}{c} \cdot \frac{u''(c) \cdot c}{u'(c)}$$

Define $\epsilon = -\frac{u''(c) \cdot c}{u'(c)} > 0$, we get the Euler equation:

$$\frac{\dot{c}}{c} = \frac{r - \rho}{\epsilon}$$

7.2 The solution of the transversality condition in 3.1

Given $\frac{\dot{\mu}(t)}{\mu(t)} = -(r - \rho)$ which is a linear non-homogeneous differential equation, we have:

$$\begin{aligned}\mu(t) &= \mu(0) \exp\left(\int_0^t (r(s) - \rho)\mathrm{d}s\right) \\ &= u'(c(0)) \exp\left(\int_0^t (r(s) - \rho)\mathrm{d}s\right)\end{aligned}$$

Substituting this expression into the TVC yields

$$\lim_{t \to \infty} \mathrm{e}^{-(\rho - n)t} a(t) u'(c(0)) \exp\left(\int_0^t (r(s) - \rho)\mathrm{d}s\right) = 0$$

Since $u'(c(0))$ can be omitted, and

$$-(\rho - n)t - \int_0^t (r(s) - \rho)\mathrm{d}s = -\int_0^t (r(s) - \rho + \rho - n)\mathrm{d}s = -\int_0^t (r(s) - n)\mathrm{d}s$$

We obtain

$$\lim_{t\to\infty} a(t)\exp\left(-\int_0^t (r(s) - n)\mathrm{d}s\right) = 0$$

7.3 The linearization of the non-linear system in 4.2

Firstly, we linearize the second equation in the neighborhood of the steady state point (k^*, c^*):

$$\begin{aligned} f(k) &\approx f(k^*) + f'(k^*)(k - k^*) \\ -(n+\delta)k &= -(n+\delta)k^* - (n+\delta)(k - k^*) \\ -c &= -c^* - (c - c^*) \end{aligned}$$

then we have

$$\begin{aligned} \dot{k} &= [f(k^*) - (n+\delta)k^* - c^*] + [f'(k^*) - (n+\delta)](k - k^*) - (c - c^*) \\ &\overset{c^*=f(k^*)-(n+\delta)k^*}{=\!=\!=\!=\!=\!=\!=} [f'(k^*) - (n+\delta)](k - k^*) - (c - c^*) \\ &\overset{f'(k^*)=\rho+\delta}{=\!=\!=\!=} (\rho - n)(k - k^*) - (c - c^*) \end{aligned}$$

For simplicity, assume the utility function is CRRA (so that ϵ is a constant). Then we linearize the first equation in the neighborhood of the steady state point (k^*, c^*):

$$\begin{aligned} \frac{1}{\epsilon}cf'(k) &\approx \frac{1}{\epsilon}c^*f'(k^*) + \frac{1}{\epsilon}\cdot 1 \cdot f'(k^*)(c - c^*) + \frac{1}{\epsilon}c^*f''(k^*)(k - k^*) \\ -(\rho+\delta)\frac{c}{\epsilon} &= -\frac{(\rho+\delta)}{\epsilon}c^* - \frac{(\rho+\delta)}{\epsilon}(c - c^*) \end{aligned}$$

then we have

$$\begin{aligned} \dot{c} &= \left[\frac{1}{\epsilon}c^*f'(k^*) - -\frac{(\rho+\delta)}{\epsilon}c^*\right] + \frac{1}{\epsilon}c^*f''(k^*)(k - k^*) + \frac{1}{\epsilon}[f'(k^*) - (\rho+\delta)](c - c^*) \\ &\overset{f'(k^*)=\rho+\delta}{=\!=\!=\!=} \frac{1}{\epsilon}c^*f''(k^*)(k - k^*) \end{aligned}$$

7.4 The analytical solution in 5.1

We use Bernoulli transformation to solve this differential equation. Specifically, define $z = \frac{y}{k} = k^{\alpha-1}$, $x = \frac{c}{k}$, the dynamic system can be written as

$$\begin{aligned}
\frac{\dot{x}}{x} &= \frac{\dot{c}}{c} - \frac{\dot{k}}{k} \\
&\xlongequal{f'(k)=\alpha k^{\alpha-1}=\alpha\frac{y}{k}} \frac{1}{\theta}\left[\alpha\frac{y}{k} - (\rho+\delta)\right] - \left(\frac{y}{k} - \frac{c}{k} - \delta\right) \\
&= \frac{1}{\theta}\left[\alpha z - (\rho+\delta)\right] - (z - x - \delta) \\
&= \left(\frac{\alpha}{\theta} - 1\right) z + x - \frac{\rho+\delta}{\theta} + \delta
\end{aligned}$$

When $\theta = \alpha$, the parameter z is eliminated, and it becomes an autonomous equation:

$$\frac{\dot{x}}{x} = x - \frac{\rho + (1-\alpha)\delta}{\alpha}$$

By separation of variables ($x > 0$), this equation can be solved. Let $B = \frac{\rho+(1-\alpha)\delta}{\alpha} > 0$, when $x \neq B$, we have

$$\mathrm{d}t = \frac{\mathrm{d}x}{x^2 - Bx} = \frac{\mathrm{d}x}{x(x-B)} = -\frac{1}{B}\left(\frac{1}{x} - \frac{1}{x-B}\right)\mathrm{d}x$$

Integrate both sides we have

$$\begin{aligned}
D_0^+ + \mathrm{e}^{Bt} &= \frac{|x-B|}{x} \\
D_1\mathrm{e}^{Bt} &= 1 - \frac{B}{x} \\
x &= \frac{B}{1 + D\mathrm{e}^{Bt}}
\end{aligned}$$

where $D \neq 0$ is a constant. Combining with the spacial case $x = B$ we have $x = \frac{B}{1+D\mathrm{e}^{Bt}}$ with $D \in \mathbb{R}$.

In order to determine the value of D, we consider the TVC:

$$\begin{aligned}
\lim_{t\to\infty} \mathrm{e}^{-\rho t}\mu(t)k(t) &= \lim_{t\to\infty} \mathrm{e}^{-\rho t}u'(c(t))k(t) \\
&= \lim_{t\to\infty} \mathrm{e}^{-\rho t}c^{-\alpha}(t)k(t) \\
&\xlongequal{x=\frac{c}{k}} \lim_{t\to\infty} \mathrm{e}^{-\rho t}x^{-\alpha}(t)k^{1-\alpha}(t) \\
&= \lim_{t\to\infty} \mathrm{e}^{-\rho t}\left(\frac{1+D\mathrm{e}^{Bt}}{B}\right)^{\alpha} k^{1-\alpha}(t) \\
&= 0
\end{aligned}$$

It is trivial that the TVC is satisfied iff $D = 0$. Therefore $x = B = \frac{\rho+(1-\alpha)\delta}{\alpha}$, and

$$c = \frac{\rho + (1-\alpha)\delta}{\alpha} k$$